## 2.6 Answers to 2.5 Exercise



**Answer 1**

$$det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

$$det\begin{pmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)(3-\lambda) - 8 = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$

$$(\lambda+1)(\lambda-5) = 0$$

$$\lambda_1 = -1 \text{ or } \lambda_2 = 5$$

$$\mathbf{D} = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$$

For $\lambda_1 = -1 : \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -1\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow y = -x \Rightarrow \boldsymbol{v}_1 = k_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}, k_1 \in \mathbb{R}$

For $\lambda_2 = 5 : \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 5\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow y = 2x \Rightarrow \boldsymbol{v}_2 = k_2\begin{pmatrix} 1 \\ 2 \end{pmatrix}, k_2 \in \mathbb{R}$

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \text{ from which } \mathbf{P}^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\mathbf{A}^k = \mathbf{P}\,\mathbf{D}^k\,\mathbf{P}^{-1}$$

$$\begin{aligned} \mathbf{A}^{100} &= \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}^{100}\frac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 5^{100} \end{pmatrix}\frac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} 5^{100}+2 & 5^{100}-1 \\ 2\times 5^{100}-2 & 2\times 5^{100}+1 \end{pmatrix} \end{aligned}$$

**[ 6 marks ]**

**Answer 2**

$$det\left(\mathbf{A} - \lambda\mathbf{I}\right) = 0$$

$$det\begin{pmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)(4-\lambda) + 2 = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 2)(\lambda - 3) = 0$$

$$\lambda_1 = 2,\ \lambda_2 = 3$$

$$\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

For $\lambda_1 = 2 : \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow y = x \Rightarrow \boldsymbol{v}_1 = k_1\begin{pmatrix} 1 \\ 1 \end{pmatrix},\ k_1 \in \mathbb{R}$

For $\lambda_2 = 3 : \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow y = 2x \Rightarrow \boldsymbol{v}_2 = k_2\begin{pmatrix} 1 \\ 2 \end{pmatrix},\ k_2 \in \mathbb{R}$

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

**[ 7 marks ]**

**Answer 3**

**( a )** $\begin{pmatrix} 1 & 0 & 4 \\ 0 & 5 & 4 \\ 4 & 4 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ which shows $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector.

The eigenvalue is $\lambda_1 = 3$

**[ 3 marks ]**

**( b )**

$$det\left(\mathbf{A} - \lambda\mathbf{I}\right) = 0$$

$$det\begin{pmatrix} 1-\lambda & 0 & 4 \\ 0 & 5-\lambda & 4 \\ 4 & 4 & 3-\lambda \end{pmatrix} = 0$$

If $\lambda_2 = 9$ is an eigenvalue it will make this equation true.

$$\begin{aligned} \text{LHS} &= det\begin{pmatrix} 1-\lambda & 0 & 4 \\ 0 & 5-\lambda & 4 \\ 4 & 4 & 3-\lambda \end{pmatrix} \\ &= det\begin{pmatrix} -8 & 0 & 4 \\ 0 & -4 & 4 \\ 4 & 4 & -6 \end{pmatrix} \\ &= (-8)\begin{vmatrix} -4 & 4 \\ 4 & -6 \end{vmatrix} + 4\begin{vmatrix} 0 & -4 \\ 4 & 4 \end{vmatrix} \\ &= (-8)(24 - 16) + 4 \times 16 \\ &= 0 \\ &= \text{RHS} \end{aligned}$$

$\therefore$ 9 is an eigenvalue $\square$

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 5 & 4 \\ 4 & 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 9\begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \left\{\begin{aligned} x + 4z &= 9x \\ 5y + 4z &= 9y \\ 4x + 4y + 3z &= 9z \end{aligned}\right. \Leftrightarrow \begin{aligned} x &= 2x \\ z &= y \\ 2x + 2y - 3z &= 0 \end{aligned}$$

Let $x = 1$ in which case, $z = 2$ and $y = 2$

The eigenvector is $k\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ where $k$ is any constant

**[ 5 marks ]**

**( c )** $\mathbf{P} = \begin{pmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \\ 1 & 2 & -2 \end{pmatrix}$

Finding the associated eigenvalue, $\lambda_3$ ...

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 5 & 4 \\ 4 & 4 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -6 \\ -3 \\ 4 \end{pmatrix} = \lambda_3\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

$$\lambda_3 = -3$$

$$\mathbf{D} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

**[ 4 marks ]**

**Answer 4**

**( a )**

$$det\left(\mathbf{A} - \lambda\mathbf{I}\right) = 0$$

$$det\begin{pmatrix} 7-\lambda & 0 & -2 \\ 0 & 5-\lambda & -2 \\ -2 & -2 & 6-\lambda \end{pmatrix} = 0$$

$$(7-\lambda)\begin{vmatrix} 5-\lambda & -2 \\ -2 & 6-\lambda \end{vmatrix} + (-2)\begin{vmatrix} 0 & 5-\lambda \\ -2 & -2 \end{vmatrix} = 0$$

$$(7-\lambda)((5-\lambda)(6-\lambda) - 4) - 2(0 + 2(5-\lambda)) = 0$$

$$\left(7-\lambda\right)\left(\lambda^2 - 11\lambda + 26\right) - 20 + 4\lambda = 0$$

$$7\lambda^2 - 77\lambda + 182 - \lambda^3 + 11\lambda^2 - 26\lambda - 20 + 4\lambda = 0$$

$$\lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0$$

As we are told that $\lambda = 9$ is a solution...

$$\begin{array}{r|rrrr} & \lambda^2 & -\;9\lambda & +\;18 & \\ \hline \lambda - 9 & \lambda^3 - 18\lambda^2 & +\;99\lambda & -\;162 & \\ & \lambda^3 - 9\lambda^2 & & & \\ \hline & -\;9\lambda^2 & +\;99\lambda & & \\ & -\;9\lambda^2 & +\;81\lambda & & \\ \hline & & 18\lambda & +\;162 & \\ & & 18\lambda & +\;162 & \\ \hline & & & 0 & \leftarrow \text{ As expected} \end{array}$$

$$\lambda^3 - 18\lambda^2 + 99\lambda - 162 = \left(\lambda - 9\right)\left(\lambda^2 - 9\lambda + 18\right)$$

$$= (\lambda - 9)(\lambda - 3)(\lambda - 6) \Leftrightarrow \lambda_1 = 3, \lambda_2 = 6, \lambda_3 = 9$$

**[ 5 marks ]**

**( b )** For $\lambda_1 = 3$,

$$\begin{pmatrix} 7 & 0 & -2 \\ 0 & 5 & -2 \\ -2 & -2 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \begin{cases} 7x - 2z = 3x \\ 5y - 2z = 3y \\ -2x - 2y + 6z = 3z \end{cases} \Leftrightarrow \quad \begin{array}{r} z = 2x \\ z = y \\ 2x + 2y - 3z = 0 \end{array}$$

Let $x = 1$ giving, $z = 2, \quad y = 2 \; : \; \boldsymbol{v}_1 = k_1\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ for any constant $k_1$

For $\lambda_2 = 6$,

$$\begin{pmatrix} 7 & 0 & -2 \\ 0 & 5 & -2 \\ -2 & -2 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 6\begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \begin{cases} 7x - 2z = 6x \\ 5y - 2z = 6y \\ -2x - 2y + 6z = 6z \end{cases} \Leftrightarrow \quad \begin{array}{r} x = 2z \\ y = -2z \\ y = -x \end{array}$$

Let $x = 2$ giving, $z = 1, y = -2 \; : \; \boldsymbol{v}_2 = k_2\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ for any constant $k_2$

For $\lambda_3 = 9$,

$$\begin{pmatrix} 7 & 0 & -2 \\ 0 & 5 & -2 \\ -2 & -2 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 9\begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \left\{\begin{aligned} 7x - 2z &= 9x \\ 5y - 2z &= 9y \\ -2x - 2y + 6z &= 9z \end{aligned}\right. \Leftrightarrow \quad \begin{aligned} z &= -x \\ z &= -2y \\ 2x + 2y + 3z &= 0 \end{aligned}$$

Let $x = 2$ giving, $z = -2$, $y = 1$ : $\boldsymbol{v}_3 = k_3\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ for any constant $k_3$

**[ 5 marks ]**

**( c )** Finding all possible scalar products pairs...

$$\boldsymbol{v}_1 \bullet \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = 1 \times 2 + 2 \times (-2) + 2 \times 1 = 0$$

$$\boldsymbol{v}_1 \bullet \boldsymbol{v}_3 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = 1 \times 2 + 2 \times 1 + 2 \times (-2) = 0$$

$$\boldsymbol{v}_2 \bullet \boldsymbol{v}_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = 2 \times 2 + (-2) \times 1 + 1 \times (-2) = 0$$

Therefore, the eigenvectors found in part (b) are mutually perpendicular. ☐

**[ 2 marks ]**

**( d )** Remembering to normalize the eigenvectors...

$$\mathbf{D} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{pmatrix} \qquad \mathbf{P} = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{pmatrix}$$

**[ 2 marks ]**

**Answer 5**

**( a )**

$$det\left( \mathbf{A} - \lambda\mathbf{I} \right) = 0$$

$$det\begin{pmatrix} 1 - \lambda & -2 & 2 \\ 0 & -3 - \lambda & 4 \\ 0 & -2 & 3 - \lambda \end{pmatrix} = 0$$

Expanding down the first column,

$$(1 - \lambda)\begin{vmatrix} -3 - \lambda & 4 \\ -2 & 3 - \lambda \end{vmatrix} = 0$$

$$(1 - \lambda)((-3 - \lambda)(3 - \lambda) + 8) = 0$$

$$\left(1 - \lambda\right)\left(\lambda^2 - 1\right) = 0$$

$$(1 - \lambda)(\lambda - 1)(\lambda + 1) = 0$$

The eigenvalues of **A** are 1, 1 and – 1

**[ 2 marks ]**

**( b )** For $\lambda_1 = -1$,

$$\begin{pmatrix} 1 & -2 & 2 \\ 0 & -3 & 4 \\ 0 & -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -1\begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \begin{cases} x - 2y + 2z = -x \\ \quad -3y + 4z = -y \\ \quad -2y + 3z = -z \end{cases} \Leftrightarrow \quad \begin{aligned} x - y + z &= 0 \\ y &= 2z \\ y &= 2z \end{aligned}$$

Let $z = 1$ giving, $y = 2, \quad x = 1 \ : \ \boldsymbol{v}_1 = k_1\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ for any constant $k_1$

For $\lambda_2 = 1$,

$$\begin{pmatrix} 1 & -2 & 2 \\ 0 & -3 & 4 \\ 0 & -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1\begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \begin{cases} x - 2y + 2z = x \\ \quad -3y + 4z = y \\ \quad -2y + 3z = z \end{cases} \Leftrightarrow \begin{aligned} y &= z \\ y &= z \\ y &= z \end{aligned}$$

The equations do not establish a relationship involving $x$ and so $x$ can be assigned an arbitrary value. The simplest pair of linearly independent eigenvectors are $\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

The set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is thus a suitable basis for $\mathbb{R}^3$

**[ 4 marks ]**

**Answer 6**

**( a )**

$$det\left(\mathbf{A} - \lambda\mathbf{I}\right) = 0$$

$$det\begin{pmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{pmatrix} = 0$$

Expanding along the top row,

$$(-\lambda)\begin{vmatrix} -\lambda & 0 \\ 1 & -\lambda \end{vmatrix} + 1\begin{vmatrix} 1 & -\lambda \\ 0 & 1 \end{vmatrix} = 0$$

$$-\lambda^3 + 1 = 0$$

$$\lambda_1 = 1,\ \lambda_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i},\ \lambda_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2}\mathbf{i}$$

**[ 4 marks ]**

**( b )** As **A** has distinct eigenvalues it is diagonizable over $\mathbb{C}$

**[ 2 marks ]**

**( c )** "The long way" will work but, looking for a short cut, ...

$$\mathbf{A}^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\mathbf{A}^3 = \mathbf{A}^2\,\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}$$

Now, $2009 = 3 \times 669 + 2$ and so,

$$\begin{aligned} \mathbf{A}^{2009} &= \mathbf{A}^{3 \times 669 + 2} \\ &= \left(\mathbf{A}^3\right)^{669}\mathbf{A}^2 \\ &= \mathbf{I}^{669}\mathbf{A}^2 \\ &= \mathbf{A}^2 \\ &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \end{aligned}$$

Matrix **A** is a permutation matrix.
A property of such a matrix is that if it is raised to a sufficient power it will equal the identity matrix.
So this short cut is not as obscure as it might at first seem !
Permutation Matrices are explored in Lecture 2 of Graph Theory I

**[ 4 marks ]**