Undergraduate Lectures in Mathematics
A First Year Course

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MatriX AlgebrA

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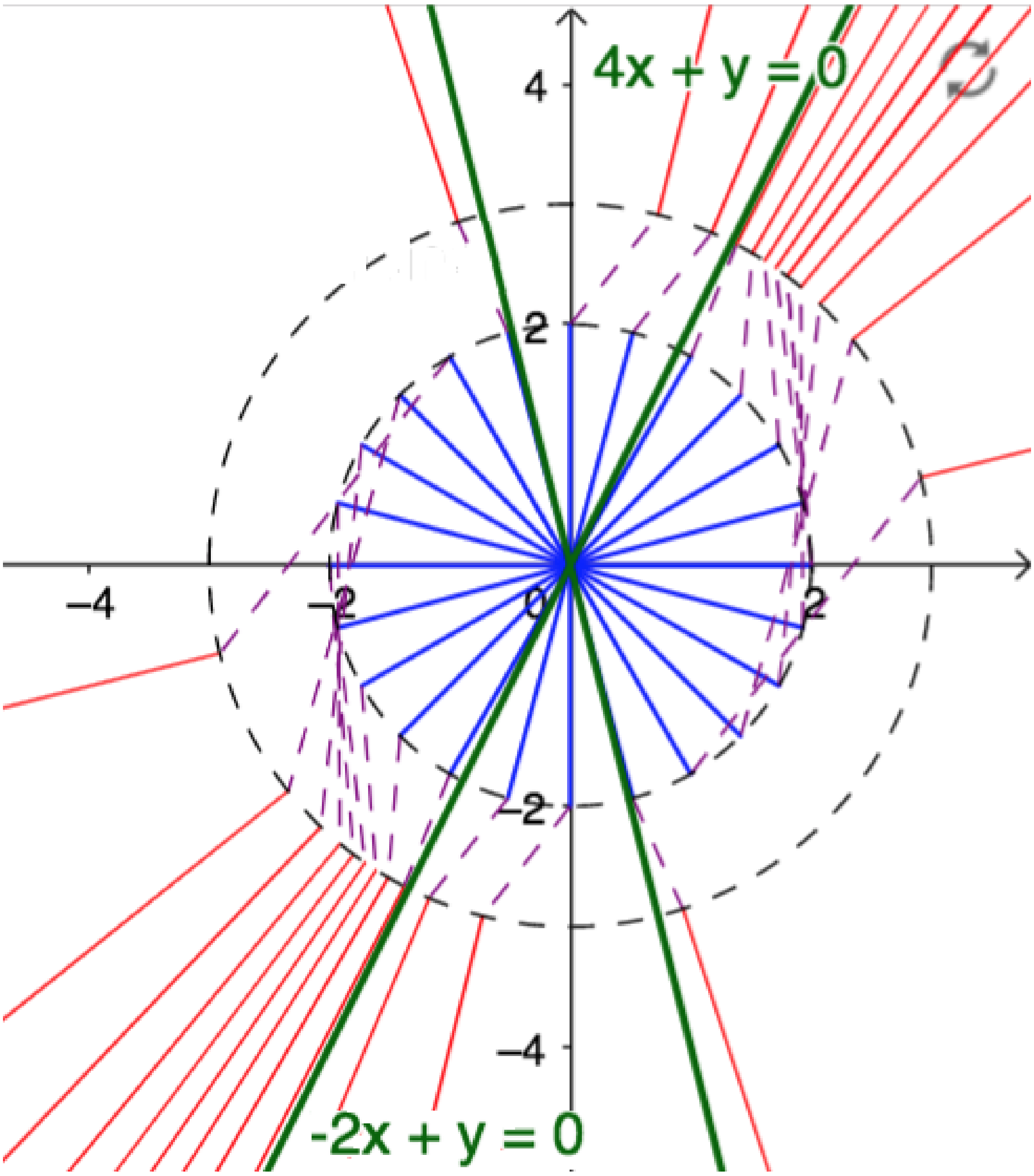


GeoGebra's "Eigenpicture" for the matrix $\begin{pmatrix} 6 & 1 \\ 8 & 4 \end{pmatrix}$ showing how lines through the origin are given varying amounts of rotation about (0, 0) by the matrix.

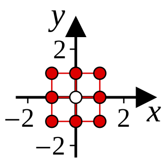
Two special invariant lines, not rotated by the matrix, are highlighted in green.

M A T R I X
A L G E B R A

Lecture 1

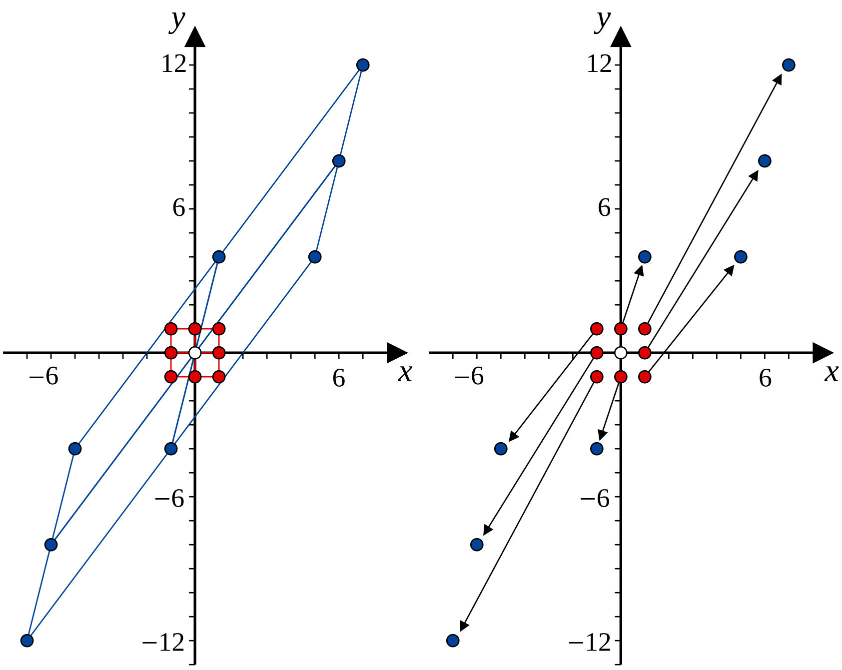
1.1 Surface Distortion : First Experiment

When a 2 × 2 matrix is applied to a point (other than the origin) it moves the point. A surface is made up of points and so it is natural to wonder about the bigger picture, namely, what is the matrix doing to the surface. To get an impression of how a matrix is distorted the surface it could be applied to a grid of four squares with corners specified by nine points.



As an example, let's apply the matrix $\mathbf{A} = \begin{pmatrix} 6 & 1 \\ 8 & 4 \end{pmatrix}$ to this grid of nine points.

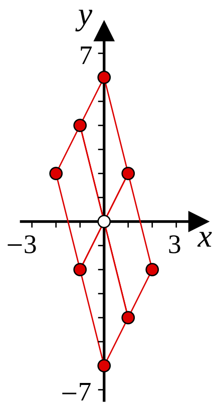
$$\begin{pmatrix} 6 & 1 \\ 8 & 4 \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} -5 & 1 & 7 & -6 & 0 & 6 & -7 & -1 & 5 \\ -4 & 4 & 12 & -8 & 0 & 8 & -12 & -4 & 4 \end{pmatrix}$$



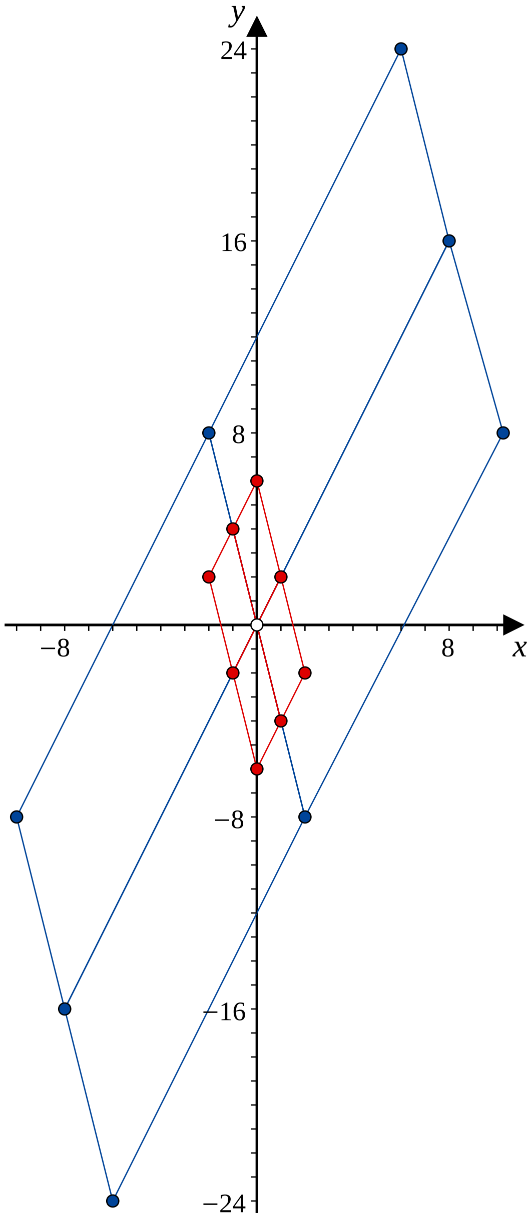
The left diagram gives an impression of how the grid has been distorted by our matrix **A**, whilst the right diagram shows how the individual points have moved. The diagrams suggest that **A** is expansive and that there is some sort of order to how the points are being moved. Clearly, something is going on but exactly what is not pinned down well by this first experiment.

1.2 Surface Distortion : Second Experiment

Progress is made by exploring other starting grids. Of those other grids the following is by far the most interesting.



This time, **A** has the following interesting effect,



This time there is an alignment between the red and the blue grids; The blue seems to be a bigger version of the red, although it is stretched more in one direction than the other. Here are the details of the calculations for this second experiment,

$$\begin{pmatrix} 6 & 1 \\ 8 & 4 \end{pmatrix} = \begin{pmatrix} -2 & -1 & 0 & -1 & 0 & 1 & 0 & 1 & 2 \\ 2 & 4 & 6 & -2 & 0 & 2 & -6 & -4 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} -10 & -2 & 6 & -8 & 0 & 8 & -6 & 2 & 10 \\ -8 & 8 & 24 & -16 & 0 & 16 & -24 & -8 & 8 \end{pmatrix}$$

1.3 Vectors and Scalings from The Experiments

In the first experiment the red grid was made up of copies of the unit vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. In the second experiment the red grid vectors are $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ which can be normalized (made of unit length) as $\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\frac{1}{\sqrt{17}}\begin{pmatrix} 1 \\ -4 \end{pmatrix}$. Typically, the red and the blue grids do not align and so the situation where they do makes the *eigenvectors* $\boldsymbol{v}_1 = k_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\boldsymbol{v}_2 = k_2\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ special (where k_1 and k_2 are scaling factors that change the vector's length but not its direction).

Focussing further on the second experiment, in the $\boldsymbol{v}_1$ direction the red point (1, 2) moved to the blue point (8, 16) which is a scaling factor of 8. This is length scaling factor is known as an *eigenvalue* λ_1. In the $\boldsymbol{v}_2$ direction the red point (1, – 4) moved to the blue point (2, – 8) which is a length scaling factor of 2. This gives another *eigenvalue* λ_2. Eigen is a German word for "special". Having seen geometrically why the eigenvectors and the eigenvalues of a matrix are very special, attention can turn to how to find them algebraically.

1.4 Eigenvalues and Eigenvectors of a Square Matrix

Definition : Eigenvalues and Eigenvectors

An eigenvector of a matrix **A** is a non-zero column vector $\boldsymbol{x}$ which satisfies the equation $\mathbf{A}\boldsymbol{x} = \lambda\boldsymbol{x}$ where λ is a scalar. The value of the scalar λ is the eigenvalue of the matrix **A** corresponding to the eigenvector $\boldsymbol{x}$.

So, for the matrix $\begin{pmatrix} 6 & 1 \\ 8 & 4 \end{pmatrix}$ we can write that,

$$\begin{pmatrix} 6 & 1 \\ 8 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 8\begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 6 & 1 \\ 8 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ -4 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

which confirms that $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector associated with eigenvalue 8, and that $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ is an eigenvector associated with eigenvalue 2.

More generally, from the definition,

$$\mathbf{A}\boldsymbol{x} = \lambda\boldsymbol{x}$$

$$\mathbf{A}x = \lambda\mathbf{I}\boldsymbol{x}$$

$$(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{x} = 0$$

As $\boldsymbol{x}$ is non-zero (by definition) the matrix $(\mathbf{A} - \lambda\mathbf{I})$ must be zero. In other words it must be singular. To be singular it must have a determinant of zero. Thus,

$$det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

This is key to finding the eigenvalues and eigenvectors of a matrix by algebra.

1.4.1 Example

Question: Use matrix algebra to find the eigenvalues and eigenvectors of $\begin{pmatrix} 6 & 1 \\ 8 & 4 \end{pmatrix}$.

Solution:

$$det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

$$det\left(\begin{pmatrix} 6 & 1 \\ 8 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = 0$$

$$det\begin{pmatrix} 6 - \lambda & 1 \\ 8 & 4 - \lambda \end{pmatrix} = 0$$

$$(6 - \lambda)(4 - \lambda) - 8 \times 1 = 0$$

$$24 - 6\lambda - 4\lambda + \lambda^2 - 8 = 0$$

$$\lambda^2 - 10\lambda + 16 = 0 \qquad \text{The characteristic equation}$$

$$(\lambda - 8)(\lambda - 2) = 0$$

$$\lambda_1 = 8 \text{ or } \lambda_2 = 2 \text{ are the eigenvalues}$$

For λ_1,

$$\begin{pmatrix} 6 & 1 \\ 8 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 8\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 6x + y \\ 8x + 4y \end{pmatrix} = \begin{pmatrix} 8x \\ 8y \end{pmatrix}$$

Matching upper elements, $6x + y = 8x \Rightarrow y = 2x \Rightarrow \boldsymbol{v}_1 = k_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

For λ_2,

$$\begin{pmatrix} 6 & 1 \\ 8 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 6x + y \\ 8x + 4y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix}$$

Matching upper elements, $6x + y = 2x \Rightarrow y = -4x \Rightarrow \boldsymbol{v}_2 = k_2\begin{pmatrix} 1 \\ -4 \end{pmatrix}$

where k_1 and k_2 are any constants.

1.5 The Cayley-Hamilton Theorem

The characteristic equation of a matrix is so called because, as we have seen, it captures some important features of the matrix. However, different matrices can have these same features in common, and so have the same characteristic equation. There is a remarkable theorem associated with the characteristic equation that is now stated without proof;

Theorem 1.1 : The Cayley-Hamilton Theorem

Every square matrix satisfies its own characteristic equation.

The theorem is now illustrated using this lecture's ongoing example, $\mathbf{A} = \begin{pmatrix} 6 & 1 \\ 8 & 4 \end{pmatrix}$.

This has the characteristic equation $\lambda^2 - 10\lambda + 16 = 0$.

The Cayley-Hamilton theorem therefore claims that $\mathbf{A}^2 - 10\mathbf{A} + 16\mathbf{I} = \mathbf{0}$.

$$
\begin{aligned}
\text{LHS} &= \mathbf{A}^2 - 10\mathbf{A} + 16\mathbf{I} \\
&= \begin{pmatrix} 6 & 1 \\ 8 & 4 \end{pmatrix}^2 - 10\begin{pmatrix} 6 & 1 \\ 8 & 4 \end{pmatrix} + 16\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 44 & 10 \\ 80 & 24 \end{pmatrix} - \begin{pmatrix} 60 & 10 \\ 80 & 40 \end{pmatrix} + \begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
&= \mathbf{0} \\
&= \text{RHS} \qquad \square
\end{aligned}
$$

1.5.1 Example

Question: Use the Cayley-Hamilton theorem to find $\begin{pmatrix} 6 & 1 \\ 8 & 4 \end{pmatrix}^{-1}$

Answer: Previously seen;

$$\mathbf{A}^2 - 10\mathbf{A} + 16\mathbf{I} = \mathbf{0}$$

$$\mathbf{A} - 10\mathbf{I} + 16\mathbf{A}^{-1} = \mathbf{0} \qquad \text{by post-multiplying through by } \mathbf{A}^{-1}$$

$$
\begin{aligned}
\mathbf{A}^{-1} &= \frac{1}{16}\left(10\mathbf{I} - \mathbf{A} \right) \\
&= \frac{1}{16}\left(10\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 6 & 1 \\ 8 & 4 \end{pmatrix} \right) \\
&= \frac{1}{16}\begin{pmatrix} 4 & -1 \\ -8 & 6 \end{pmatrix}
\end{aligned}
$$

1.6 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 50

Question 1

The linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ maps the point (5, 2) to the point ($-$ 15, $-$ 6).

(i) Write down an eigenvalue of the matrix representing T.

[1 mark]

(ii) Write down a corresponding eigenvector.

[2 marks]

(iii) Write down the equation of the corresponding invariant line.

[1 mark]

Question 2

Show that any 2×2 matrix of the form $\mathbf{A} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, $a, b \in \mathbb{R}$ has complex eigenvalues given by $a \pm b\mathbf{i}$

[4 marks]

Question 3

The matrix $\mathbf{A} = \begin{pmatrix} 3 & k \\ 1 & -1 \end{pmatrix}$ has a repeated eigenvalue.

(i) Find the value of k.

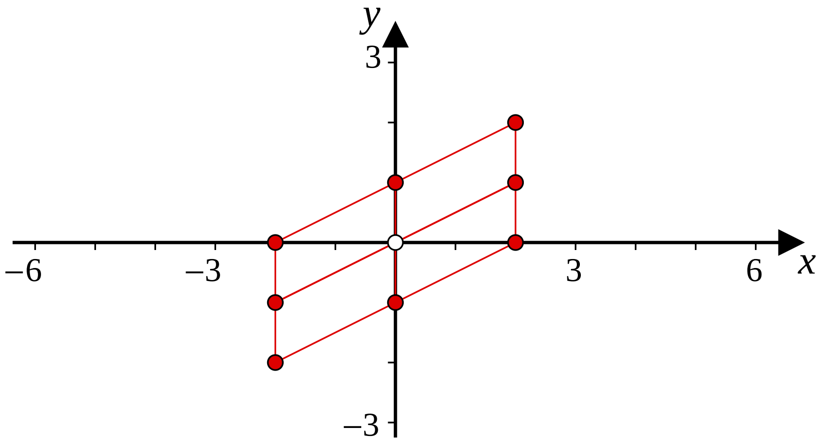
[4 marks]

(ii) Find the value of the repeated eigenvalue, an equation of a corresponding invariant line and a corresponding eigenvector.

[3 marks]

(iii) The red grid below is specified by $\begin{pmatrix} -2 & 0 & 2 & -2 & 0 & 2 & -2 & 0 & 2 \\ 0 & 1 & 2 & -1 & 0 & 1 & -2 & -1 & 0 \end{pmatrix}$.

Draw the blue grid that results then transforming this by $\mathbf{A}$.



[4 marks]

(iv) Make two comments on your part (iii) answer.

[2 marks]

Question 4

A-Level Examination Question from May 2019, Paper FP2, Q1 (Edexcel)

Given that,

$$\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}$$

(a) find the characteristic equation for the matrix **A**.
Simplify your answer.

[2 marks]

(b) Hence find an expression for the matrix $\mathbf{A}^{-1}$ in the form $\lambda\mathbf{A} + \mu\mathbf{I}$,
where λ and μ are constants to be found.

[3 marks]

Question 5

Let λ be an eigenvalue of a matrix **A**

Prove by induction that λ^n is an eigenvalue of $\mathbf{A}^n$, for all $n \in \mathbb{Z}^+$

[6 marks]

Question 6

A-Level Examination Question from June 2020, Paper FP2, Q3 (Edexcel)

$$\mathbf{M} = \begin{pmatrix} 1 & k & -2 \\ 2 & -4 & 1 \\ 1 & 2 & 3 \end{pmatrix} \quad \text{where } k \text{ is a constant}$$

(a) Show that, in terms of k, a characteristic equation for $\mathbf{M}$ is given by,

$$\lambda^3 - (2k + 13)\lambda + 5(k + 6) = 0$$

[3 marks]

Given that $det\,\mathbf{M} = 5$

(b) **(i)** find the value of k

(ii) use the Cayley-Hamilton theorem to find the inverse of $\mathbf{M}$

[7 marks]

Question 7

Show that any 2×2 matrix of the form $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfies its own characteristic equation and hence prove the Cayley-Hamilton theorem for 2×2 matrices.

[8 marks]

1.7 Answers to 1.6 Exercise

Answer 1

(i) Crucially, the displacement vector to the first point is a multiple of the displacement vector to the second point and so the two points lie on a line that passes through the origin.
The length scaling factor between these displacement vectors gives an eigenvalue, λ, of the matrix representing the transformation; $\lambda = -3$

[1 mark]

(ii) $\boldsymbol{v} \;=\; k\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ for any real constant k

[2 marks]

(iii) $y \;=\; \dfrac{2}{5}\,x$ (The line through the origin on which both points lie)

[1 mark]

Answer 2

$$\begin{aligned}
det\,(\,\mathbf{A} - \lambda\mathbf{I}\,) &= 0 \\
det\begin{pmatrix} a - \lambda & b \\ -b & a - \lambda \end{pmatrix} &= 0 \\
(\,a - \lambda\,)(\,a - \lambda\,) + b^2 &= 0 \\
\lambda^2 - 2a\lambda + a^2 + b^2 &= 0
\end{aligned}$$

$$\begin{aligned}
\lambda &= \frac{2a \pm \sqrt{4\,a^2 - 4\,(\,a^2 + b^2\,)}}{2} \\
&= \frac{2a \pm \sqrt{-4\,b^2}}{2} \\
&= a \pm b\mathbf{i} \qquad \square
\end{aligned}$$

[4 marks]

Answer 3

(i) For a repeated eigenvalue the discriminant of the characteristic equation must be zero. $\quad det\,(\,\mathbf{A} - \lambda\mathbf{I}\,) \;=\; 0$

$$\begin{aligned}
det\left(\begin{pmatrix} 3 - \lambda & k \\ 1 & -1 - \lambda \end{pmatrix}\right) &= 0 \\
(\,3 - \lambda\,)(-1 - \lambda\,) - k &= 0 \\
\lambda^2 - 2\lambda - (\,3 + k\,) &= 0 \qquad \text{The characteristic equation} \\
(-2)^2 + 4\,(\,3 + k\,) &= 0 \qquad \text{working on the discriminant} \\
k &= -4
\end{aligned}$$

[4 marks]

(ii) The characteristic equation with $k = -4$ becomes $\lambda^2 - 2\lambda + 1 = 0$

from which $(\lambda - 1)^2 = 0$ giving $\lambda = 1$

For the corresponding eigenvector, $\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow y = \frac{1}{2}x$

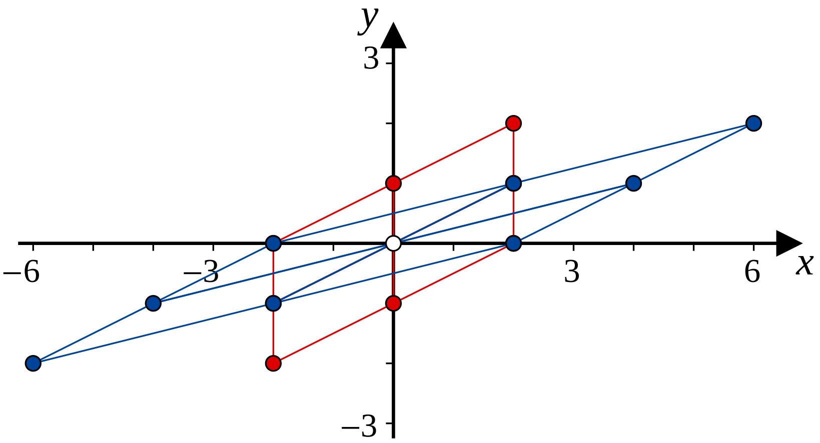
$$\boldsymbol{v} = \mu\begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ for any real constant } \mu$$

[3 marks]

(iii)

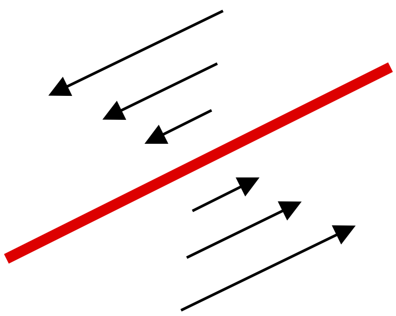
$$\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -2 & 0 & 2 & -2 & 0 & 2 & -2 & 0 & 2 \\ 0 & 1 & 2 & -1 & 0 & 1 & -2 & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -6 & -4 & -2 & -2 & 0 & 2 & 2 & 4 & 6 \\ -2 & -1 & 0 & -1 & 0 & 1 & 0 & 1 & 2 \end{pmatrix}$$



[4 marks]

(iv) The eigenvalue of 1 along the line $y = \frac{1}{2}x$ means that points on that line do not move. There are no other invariant points or line.
There is a shearing action either side of the point invariant line that increases linearly with the perpendicular distance away from the line.
This can be depicted geometrically as;



[2 marks]

Answer 4

(a)

$$det (\mathbf{A} - \lambda\mathbf{I}) = 0$$

$$det\left(\begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = 0$$

$$det\begin{pmatrix} 3 - \lambda & 2 \\ 2 & 2 - \lambda \end{pmatrix} = 0$$

$$(3 - \lambda)(2 - \lambda) - 4 = 0$$

$$\lambda^2 - 5\lambda + 2 = 0$$

[2 marks]

(b) The Cayley-Hamilton theorem states that every square matrix satisfies its own characteristic equation.

$$\mathbf{A}^2 - 5\mathbf{A} + 2 = 0$$

$$\mathbf{A} - 5\mathbf{I} + 2\mathbf{A}^{-1} = 0 \qquad \text{by post-multiplying through by } \mathbf{A}^{-1}$$

$$2\mathbf{A}^{-1} = -\mathbf{A} + 5\mathbf{I}$$

$$\mathbf{A}^{-1} = -\frac{1}{2}\mathbf{A} + \frac{5}{2}\mathbf{I}$$

That is, $\lambda = -\dfrac{1}{2}$ and $\mu = \dfrac{5}{2}$

[3 marks]

Answer 5

Statement: The matrix $\mathbf{A}^n$ has eigenvalue λ^n where λ is an eigenvalue of $\mathbf{A}$

Proof by induction: Keep in mind the definition of an eigenvalue;

Definition : Eigenvalues and Eigenvectors
An eigenvector of a matrix $\mathbf{A}$ is a non-zero column vector $\boldsymbol{x}$ which satisfies the equation $\mathbf{A}\boldsymbol{x} = \lambda\boldsymbol{x}$ where λ is a scalar. The value of the scalar λ is the eigenvalue of the matrix $\mathbf{A}$ corresponding to the eigenvector $\boldsymbol{x}$.

When $n = 1$, this is true by the above definition.
Assume that for some $n = k$ the statement is true.

That is $\mathbf{A}^k$ has eigenvalue λ^k in which case $\mathbf{A}^k\boldsymbol{x} = \lambda^k\boldsymbol{x}$ is assumed true.
Front-multiply both sides of this by $\mathbf{A}$;

$$\mathbf{A}\left(\mathbf{A}^k\boldsymbol{x}\right) = \mathbf{A}\left(\lambda^k\boldsymbol{x}\right)$$

$$\mathbf{A}^{k+1}\boldsymbol{x} = \lambda^k(\mathbf{A}\boldsymbol{x}) \qquad \text{as } \lambda \text{ is a scalar and so commutative}$$

$$\mathbf{A}^{k+1}\boldsymbol{x} = \lambda^k(\lambda\boldsymbol{x}) \qquad \text{by the definition}$$

$$\mathbf{A}^{k+1}x = \lambda^{k+1}x$$

which shows that if it is true that λ^k is an eigenvalue of $\mathbf{A}^k$ then λ^{k+1} is an eigenvalue of A^{k+1}. So if true for $n = k$ then the statement is true for $n = k + 1$ and, as it was true for $n = 1$, by mathematical induction it is true for all $n \in \mathbb{Z}^+$

[6 marks]

Answer 6

(a)

$$det\left(\mathbf{A} - \lambda\mathbf{I}\right) = 0$$

$$\begin{vmatrix} 1-\lambda & k & -2 \\ 2 & -4-\lambda & 1 \\ 1 & 2 & 3-\lambda \end{vmatrix} = 0$$

Expanding along the top row...

$$(1-\lambda)((-4-\lambda)(3-\lambda)-2) - k(2(3-\lambda)-1) - 2(4-(-4-\lambda)) = 0$$

$$\left(1-\lambda\right)\left(\lambda^2+\lambda-14\right) - k\left(5-2\lambda\right) - 2\left(8+\lambda\right) = 0$$

$$\lambda^2 + \lambda - 14 - \lambda^3 - \lambda^2 + 14\lambda - 5k + 2\lambda k - 16 - 2\lambda = 0$$

$$-\lambda^3 + \lambda(13+2k) - 30 - 5k = 0$$

$$\lambda^3 - \lambda(2k+13) + 5(k+6) = 0 \qquad \square$$

[3 marks]

(b) **(i)** $\begin{vmatrix} 1-\lambda & k & -2 \\ 2 & -4-\lambda & 1 \\ 1 & 2 & 3-\lambda \end{vmatrix}$ with $\lambda = 0$ becomes $\begin{vmatrix} 1 & k & -2 \\ 2 & -4 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

so solving $\begin{vmatrix} 1 & k & -2 \\ 2 & -4 & 1 \\ 1 & 2 & 3 \end{vmatrix} = 5$

is the same as solving $-5(k+6) = 5$

(The minus sign is undoing the sign flip in the last step of part (a))

$$\therefore\ k = -7$$

[2 marks]

(ii) By the Cayley-Hamilton theorem, $\mathbf{M}^3 + \mathbf{M} - 5\mathbf{I} = \mathbf{0}$

Post-multiplying by $\mathbf{M}^{-1}$ gives $\mathbf{M}^2 + \mathbf{I} - 5\mathbf{M}^{-1} = \mathbf{0}$

$$\mathbf{M}^{-1} = \frac{1}{5}\left(\mathbf{M}^2 + \mathbf{I}\right)$$

$$= \frac{1}{5}\left(\begin{pmatrix} 1 & -7 & -2 \\ 2 & -4 & 1 \\ 1 & 2 & 3 \end{pmatrix}^2 + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)$$

$$= \frac{1}{5}\left(\begin{pmatrix} -15 & 17 & -15 \\ -5 & 4 & -5 \\ 8 & -9 & 9 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)$$

$$= \frac{1}{5}\begin{pmatrix} -14 & 17 & -15 \\ -5 & 5 & -5 \\ 8 & -9 & 10 \end{pmatrix}$$

[5 marks]

Answer 7

Finding the generalised characteristic equation...

$$det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

$$det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$$

$$det\begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = 0$$

$$(a - \lambda)(d - \lambda) - bc = 0$$

$$\lambda^2 - (a + d)\lambda + ad - bc = 0$$

The Cayley-Hamilton theorem now claims that,

$$\mathbf{M}^2 - (a + d)\mathbf{M} + (ad - bc)\mathbf{I} = \mathbf{0}$$

$$\begin{aligned}
\text{LHS} &= \mathbf{M}^2 - (a + d)\mathbf{M} + (ad - bc)\mathbf{I} \\
&= \begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 - (a + d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} + (ad - bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix} - \begin{pmatrix} a^2 + ad & ab + bd \\ ac + cd & ad + d^2 \end{pmatrix} + \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} \\
&= \begin{pmatrix} a^2 + bc - a^2 - ad + ad - bc & ab + bd - ab - bd + 0 \\ ac + cd - ac - cd + 0 & bc + d^2 - ad - d^2 + ad - bc \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
&= \mathbf{0} \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[8 marks]

2.1 Diagonal Matrices

In the first lecture, the focus was upon general square matrices. This time we specialise by looking at a category of square matrix that has interesting properties; the diagonal matrix.

Definition : A Diagonal Matrix

A diagonal matrix is a square matrix in which all elements not on the leading diagonal are zero.

It is of the form $\begin{pmatrix} a_{11} & 0 & 0 & \ldots & 0 \\ 0 & a_{22} & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & 0 & a_{nn} \end{pmatrix}$ where $a_{ij} = 0$ if $i \neq j$

The initial interest in diagonal matrices arises from the following simple observation;

For a diagonal matrix $\mathbf{D} = \begin{pmatrix} a_{11} & 0 & 0 & \ldots & 0 \\ 0 & a_{22} & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & 0 & a_{nn} \end{pmatrix}$, $\mathbf{D}^k = \begin{pmatrix} a_{11}^k & 0 & 0 & \ldots & 0 \\ 0 & a_{22}^k & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & 0 & a_{nn}^k \end{pmatrix}$

So, for example, $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^4 = \begin{pmatrix} 2^4 & 0 \\ 0 & 3^4 \end{pmatrix} = \begin{pmatrix} 16 & 0 \\ 0 & 81 \end{pmatrix}$

By itself, this is unremarkable. However, many other matrices, $\mathbf{A}$, can be written in the form $\mathbf{A} = \mathbf{P}\ \mathbf{D}\ \mathbf{P}^{-1}$ for some matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$. This suggests a possible shortcut to working out $\mathbf{A}^k$ because,

$$\begin{aligned} \mathbf{A}^k &= \left(\mathbf{PD}\,\mathbf{P}^{-1}\right)^k \\ &= \left(\mathbf{PD}\,\mathbf{P}^{-1}\right)\left(\mathbf{PD}\,\mathbf{P}^{-1}\right)\ldots\left(\mathbf{PD}\,\mathbf{P}^{-1}\right) \\ &= \mathbf{PD}\left(\mathbf{P}^{-1}\,\mathbf{P}\right)\mathbf{D}\left(\mathbf{P}^{-1}\,\mathbf{P}\right)\ldots\left(\mathbf{P}^{-1}\,\mathbf{P}\right)\mathbf{D}\,\mathbf{P}^{-1} \\ &= \mathbf{P}\,\mathbf{D}^k\,\mathbf{P}^{-1} \end{aligned}$$

For this to be worthwhile, given a matrix $\mathbf{A}$, it would need to be relatively straight forward to find the corresponding matrices $\mathbf{P}$, $\mathbf{D}$ and $\mathbf{P}^{-1}$ ahead of using the fact that $\mathbf{A}^k = \mathbf{P}\,\mathbf{D}^k\,\mathbf{P}^{-1}$

2.2 How to Find P and D

Given a $n \times n$ square matrix **A** that is diagonalizable, **D** and **P** are found by;

Step 1: Find the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ of the matrix **A**
(If there are n distinct eigenvalues, **A** is diagonalizable)

Step 2: $\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & 0 & ... & 0 \\ 0 & \lambda_2 & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & ... & 0 & 0 & \lambda_n \end{pmatrix}$ where $a_{ij} = 0$ if $i \neq j$, $a_{ij} = \lambda_i$ if $i = j$

Step 3: Find the corresponding eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n$ of the matrix A

Step 4: $\mathbf{P} = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & ... & \boldsymbol{v}_n \end{pmatrix}$

2.2.1 Example

Question: Find the matrices **P** and **D** given that $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}$ and hence $\mathbf{A}^{100}$

Answer:

$$det(\mathbf{A} - \lambda \mathbf{I}) = 0$$

$$det\begin{pmatrix} 3-\lambda & -1 \\ 2 & -\lambda \end{pmatrix} = 0$$

$$(3-\lambda)(-\lambda) + 2 = 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda - 1)(\lambda - 2) = 0$$

$$\lambda_1 = 1 \text{ or } \lambda_2 = 2$$

$$\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

For $\lambda_1 = 1 : \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow y = 2x \Rightarrow \boldsymbol{v}_1 = k_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \ k_1 \in \mathbb{R}$

For $\lambda_2 = 2 : \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow y = x \Rightarrow \boldsymbol{v}_2 = k_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ k_2 \in \mathbb{R}$

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \text{ from which } \mathbf{P}^{-1} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$$

$$\mathbf{A}^k = \mathbf{P}\,\mathbf{D}^k\,\mathbf{P}^{-1}$$

$$\mathbf{A}^{100} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^{100}\begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2^{100} \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^{101} - 1 & 1 - 2^{100} \\ 2^{101} - 2 & 2 - 2^{100} \end{pmatrix}$$

2.3 Understanding Diagonalization

To recap, so far it has been observed that $n \times n$ matrices with n distinct eigenvalues, **A**, can be written in the form $\mathbf{A} = \mathbf{P}\,\mathbf{D}\,\mathbf{P}^{-1}$ where **D** is a diagonal matrix. The motivation to do this was to find a shortcut method to calculate powers of **A** by using the deduced result that,

$$\mathbf{A}^k = \mathbf{P}\,\mathbf{D}^k\,\mathbf{P}^{-1}$$

However, the fact that **A** has this matrix **D** associated with it is interesting in its own right. To better understand this association a specific example will be considered, this time with an eye upon the geometry it represents.

Our fresh example will feature the matrix $\mathbf{A} = \begin{pmatrix} 11 & -2 \\ -2 & 14 \end{pmatrix}$

Its eigenvalues are found in the established fashion as follows;

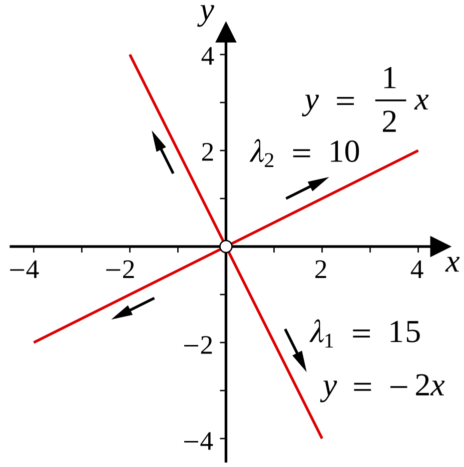
$$\begin{aligned} det\left(\mathbf{A} - \lambda \mathbf{I} \right) &= 0 \\ det\begin{pmatrix} 11 - \lambda & -2 \\ -2 & 14 - \lambda \end{pmatrix} &= 0 \\ (11 - \lambda)(14 - \lambda) - 4 &= 0 \\ \lambda^2 - 25\lambda + 150 &= 0 \\ (\lambda - 15)(\lambda - 10) &= 0 \\ \lambda_1 = 15 \text{ or } \lambda_2 &= 10 \end{aligned}$$

In deducing the eigenvectors, this time they will be normalized. By making them have a length of unity, the matrix **P** and its inverse will be easier to interpret geometrically.

For $\lambda_1 = 15$: $\begin{pmatrix} 11 & -2 \\ -2 & 14 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 15\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow y = -2x \Rightarrow \|\mathbf{v}_1\| = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

For $\lambda_2 = 10$: $\begin{pmatrix} 11 & -2 \\ -2 & 14 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 10\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow y = \frac{1}{2}x \Rightarrow \|\mathbf{v}_2\| = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Geometrically, the following diagram shows the two invariant lines along which the eigenvalues give the length scaling factor.



The matrices **P**, **D** and $\mathbf{P}^{-1}$ can now be written down following the steps described previously,

$$\mathbf{P} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 15 & 0 \\ 0 & 10 \end{pmatrix} \text{ and } \mathbf{P}^{-1} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

Geometrically, D is a scaling matrix that scales with a length scale factor of 15 along the *x*-axis and 10 along the *y*, easily seen from the following calculation,

$$\begin{pmatrix} 15 & 0 \\ 0 & 10 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 15x \\ 10y \end{pmatrix}$$

A rotation matrix has the form $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and, amazingly, both **P** and $\mathbf{P}^{-1}$ are such matrices. For $\mathbf{P}^{-1}$, $\cos\theta = \frac{1}{\sqrt{5}}$ and $\sin\theta = \frac{2}{\sqrt{5}}$ and the angle of (anticlockwise) rotation is given by

$$\begin{aligned} \tan\theta &= \frac{\sin\theta}{\cos\theta} \\ &= \frac{2}{1} \\ \theta &= 63.4° \end{aligned}$$

A similar calculation shows that **P** is a same angle rotation but in the opposite direction, clockwise.

The matrix decomposition **A** = **P D** $\mathbf{P}^{-1}$ in this example is thus saying that,

- A scaling of 15 along $y = -2x$ and 10 along $y = \frac{1}{2}x$

can also be achieved by,

- Rotating the lines $y = -2x$ and $y = \frac{1}{2}x$ by $63.4°$ $\qquad \mathbf{P}^{-1}$

  (such that they lie along the *x*-axis and the *y*-axis respectively)
- Scaling by 15 along the *x*-axis and by 10 along the *y*-axis $\qquad$ **D**
- Rotating the *x* and *y*-axis back onto $y = -2x$ and $y = \frac{1}{2}x$ $\qquad$ **P**

Notice that the matrices **A** and **D** have the same characteristic equation.

Our example covered a most useful special case (that will be investigated further next) in which the invariant lines along which the eigenvectors and eigenvalues acted were mutually perpendicular. In the more general case, where this is not so, the matrix **P** has to do more than just rotate; it has to adjust the angle between the invariant lines to map them onto the *x* and *y* axes. This makes the geometric interpretation more involved. Exploring such is left as an exercise for the interested reader !

2.4 The Orthonormal Matrix

The section 2.3 example involved a matrix **A** that was orthogonal. To explain what that is, a detour is required involving the transpose matrix, the symmetric matrix, orthogonal vectors and orthonormal vectors. However, orthogonal matrices have a marvellous property that will make the journey worthwhile.

Transpose of a Matrix → Symmetric Matrix → Orthonormal Vectors → Orthogonal Matrix

---

Definition : The Transpose of an $n \times n$ Matrix

Given the matrix $\mathbf{M} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$ the transpose of matrix **M**

is denoted $\mathbf{M}^{\mathrm{T}}$ and is formed by an interchange of rows and columns.

Thus, $$\mathbf{M}^{\mathrm{T}} = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{pmatrix}$$

For example, $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$

Definition : A Symmetric Matrix

A matrix, **M**, is symmetric if $\mathbf{M} = \mathbf{M}^{\mathrm{T}}$. Such matrices are readily recognised for their elements are symmetric with respect to the leading diagonal.

For example, $\begin{pmatrix} a & b & c \\ b & e & f \\ c & f & g \end{pmatrix}$ is symmetric (about its leading diagonal shown red)

Two vectors are described as being orthogonal if they are perpendicular to each other. One way to test two (non-zero) vectors to determine if they are orthogonal is to calculate their scalar (dot) product. If that is zero then the the vectors are orthogonal, otherwise not. A set of vectors $\{ \boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n \}$ are mutually orthogonal if every pair of vectors is orthogonal. In other words if,

$$\boldsymbol{v}_i \bullet \boldsymbol{v}_j = 0 \text{ for all } i \neq j$$

Example: To show the vectors $\begin{pmatrix} \sqrt{2} \\ 0 \\ -\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$ are mutually orthogonal work through the three possible scalar products showing they are all zero.

$$\begin{pmatrix} \sqrt{2} \\ 0 \\ -\sqrt{2} \end{pmatrix} \bullet \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} = \sqrt{2} \times 1 + 0 \times \sqrt{2} + \left(-\sqrt{2}\right) \times 1 = 0$$

$$\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} = 1 \times 1 + \sqrt{2} \times \left(-\sqrt{2}\right) + 1 \times 1 = 0$$

$$\begin{pmatrix} \sqrt{2} \\ 0 \\ -\sqrt{2} \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} = \sqrt{2} \times 1 + 0 \times \left(-\sqrt{2}\right) + \left(-\sqrt{2}\right) \times 1 = 0$$

As all scalar products are zero the three vectors are mutually orthogonal. ☐

If a set of mutually orthogonal vectors are normalized they are termed a set of orthonormal vectors. Normalizing a vector means keeping its direction but scaling it so that it has a length of 1. This is done by dividing the vector by its magnitude,

$$\|\boldsymbol{v}\| = \frac{\boldsymbol{v}}{|\boldsymbol{v}|}$$

where $\|\boldsymbol{v}\|$ denotes a vector $\boldsymbol{v}$, normalized, and $|\boldsymbol{v}|$ is the vector's magnitude.

The vectors of the previous example, normalized, become,

$$\left\|\begin{pmatrix} \sqrt{2} \\ 0 \\ -\sqrt{2} \end{pmatrix}\right\| = \frac{1}{2}\begin{pmatrix} \sqrt{2} \\ 0 \\ -\sqrt{2} \end{pmatrix}, \quad \left\|\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}\right\| = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}, \quad \left\|\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}\right\| = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

Definition : An Orthogonal Matrix

Given a set of n orthonormal n-dimensional vectors, $\{v_1, v_2, ..., v_n\}$, the $n \times n$ square matrix $\mathbf{M} = \left(\boldsymbol{v}_1 \ \boldsymbol{v}_2 \ \dots \ \boldsymbol{v}_n \right)$ with real entries is termed an orthogonal matrix. An orthogonal matrix, M, has the marvellous property that,

$$\mathbf{M}^{\mathrm{T}}\mathbf{M} = \mathbf{I} = \mathbf{M}\,\mathbf{M}^{\mathrm{T}} \Leftrightarrow \mathbf{M}^{-1} = \mathbf{M}^{\mathrm{T}}$$

Working out the inverse of an $n \times n$ matrix for $n > 2$ can be hard work but for an orthogonal matrix is easy as it is simply the transpose.

Thus, building on the previous example, the following matrix, **M**, is orthogonal and has an inverse that is simply its transpose.

$$\mathbf{M} = \frac{1}{2}\begin{pmatrix} \sqrt{2} & 1 & 1 \\ 0 & \sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & 1 & 1 \end{pmatrix}, \quad \mathbf{M}^{-1} = \mathbf{M}^{\mathrm{T}} = \frac{1}{2}\begin{pmatrix} \sqrt{2} & 0 & -\sqrt{2} \\ 1 & \sqrt{2} & 1 \\ 1 & -\sqrt{2} & 1 \end{pmatrix}$$

The reader may like to check that, for this example, $\mathbf{M}\,\mathbf{M}^{\mathrm{T}} = \mathbf{I}$

2.5 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 50

Question 1

Decompose the matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ into the form $\mathbf{P}\,\mathbf{D}\,\mathbf{P}^{-1}$ where $\mathbf{D}$ is a diagonal matrix and hence determine $\mathbf{A}^{100}$ leaving the entries in index form.

[6 marks]

Question 2

A-Level Examination Question from May 2018, Paper FPM2, Q4 (Edexcel)

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$$

Find a matrix **P** and a diagonal matrix **D** such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\,\mathbf{P}$

[7 marks]

Question 3

A-Level Sample Assessment Materials from 2018, Paper FPM2, Q2 (Edexcel)

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 5 & 4 \\ 4 & 4 & 3 \end{pmatrix}$$

(a) Verify $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector of **A** and find the corresponding eigenvalue.

[3 marks]

(b) Show 9 is an eigenvalue of **A** and find the corresponding eigenvector.

[5 marks]

Given that a third eigenvector of **A** is $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

(c) write down a matrix **P** and a diagonal matrix **D** such that $\mathbf{P}^{-1}\mathbf{A}\,\mathbf{P} = \mathbf{D}$

[4 marks]

Question 4

The matrix $\mathbf{A} = \begin{pmatrix} 7 & 0 & -2 \\ 0 & 5 & -2 \\ -2 & -2 & 6 \end{pmatrix}$

(a) Given 9 is an eigenvalue, find the other two eigenvalues of **A**.

[5 marks]

(b) Find the eigenvectors corresponding to the eigenvalues of **A**.

[5 marks]

(c) Show the eigenvectors found in part (b) are mutually perpendicular.

[2 marks]

(d) Find a matrix **P** and a diagonal matrix **D** such that $\mathbf{P}^{T}\ \mathbf{A}\ \mathbf{P} = \mathbf{D}$

[2 marks]

Question 5

Open University (UK) Specimen Examination Question from 1994, M203, Q4

The linear transformation $f : \mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix,

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 2 \\ 0 & -3 & 4 \\ 0 & -2 & 3 \end{pmatrix}$$

(a) Find the eigenvalues of **A** with respect to the standard basis.

[2 marks]

(b) Find a basis for $\mathbb{R}^3$ consisting of eigenvectors of **A**.
(That is, find corresponding eigenvectors)

[4 marks]

Question 6

Princeton University Examination Question, Linear Algebra Paper, 2009

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

(a) Find the characteristic polynomial and all the eigenvalues (real and complex) of **A**.

[4 marks]

(b) Giving a reason, state if **A** is diagonizable over the complex numbers.

[2 mark]

(c) Calculate $\mathbf{A}^{2009}$

[4 marks]

2.6 Answers to 2.5 Exercise

Answer 1

$$det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

$$det\begin{pmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)(3-\lambda) - 8 = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$

$$(\lambda+1)(\lambda-5) = 0$$

$$\lambda_1 = -1 \text{ or } \lambda_2 = 5$$

$$\mathbf{D} = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$$

For $\lambda_1 = -1 : \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -1\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow y = -x \Rightarrow \boldsymbol{v}_1 = k_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}, k_1 \in \mathbb{R}$

For $\lambda_2 = 5 : \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 5\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow y = 2x \Rightarrow \boldsymbol{v}_2 = k_2\begin{pmatrix} 1 \\ 2 \end{pmatrix}, k_2 \in \mathbb{R}$

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \text{ from which } \mathbf{P}^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\mathbf{A}^k = \mathbf{P}\,\mathbf{D}^k\,\mathbf{P}^{-1}$$

$$\mathbf{A}^{100} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}^{100}\frac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 5^{100} \end{pmatrix}\frac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 5^{100} + 2 & 5^{100} - 1 \\ 2 \times 5^{100} - 2 & 2 \times 5^{100} + 1 \end{pmatrix}$$

[6 marks]

Answer 2

$$det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

$$det\begin{pmatrix} 1 - \lambda & 1 \\ -2 & 4 - \lambda \end{pmatrix} = 0$$

$$(1 - \lambda)(4 - \lambda) + 2 = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 2)(\lambda - 3) = 0$$

$$\lambda_1 = 2, \ \lambda_2 = 3$$

$$\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

For $\lambda_1 = 2 : \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow y = x \Rightarrow \boldsymbol{v}_1 = k_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}, k_1 \in \mathbb{R}$

For $\lambda_2 = 3 : \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow y = 2x \Rightarrow \boldsymbol{v}_2 = k_2\begin{pmatrix} 1 \\ 2 \end{pmatrix}, k_2 \in \mathbb{R}$

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

[7 marks]

Answer 3

(a) $\begin{pmatrix} 1 & 0 & 4 \\ 0 & 5 & 4 \\ 4 & 4 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ which shows $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector.

The eigenvalue is $\lambda_1 = 3$

[3 marks]

(b)

$$det\left(\mathbf{A} - \lambda\mathbf{I}\right) = 0$$

$$det\begin{pmatrix} 1-\lambda & 0 & 4 \\ 0 & 5-\lambda & 4 \\ 4 & 4 & 3-\lambda \end{pmatrix} = 0$$

If $\lambda_2 = 9$ is an eigenvalue it will make this equation true.

$$\begin{aligned} \text{LHS} &= det\begin{pmatrix} 1-\lambda & 0 & 4 \\ 0 & 5-\lambda & 4 \\ 4 & 4 & 3-\lambda \end{pmatrix} \\ &= det\begin{pmatrix} -8 & 0 & 4 \\ 0 & -4 & 4 \\ 4 & 4 & -6 \end{pmatrix} \\ &= (-8)\begin{vmatrix} -4 & 4 \\ 4 & -6 \end{vmatrix} + 4\begin{vmatrix} 0 & -4 \\ 4 & 4 \end{vmatrix} \\ &= (-8)(24 - 16) + 4 \times 16 \\ &= 0 \\ &= \text{RHS} \end{aligned}$$

$\therefore$ 9 is an eigenvalue $\square$

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 5 & 4 \\ 4 & 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 9\begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \begin{cases} x + 4z = 9x \\ 5y + 4z = 9y \\ 4x + 4y + 3z = 9z \end{cases} \Leftrightarrow \quad \begin{aligned} x &= 2x \\ z &= y \\ 2x + 2y - 3z &= 0 \end{aligned}$$

Let $x = 1$ in which case, $z = 2$ and $y = 2$

The eigenvector is $k\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ where k is any constant

[5 marks]

(c) $\mathbf{P} = \begin{pmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \\ 1 & 2 & -2 \end{pmatrix}$

Finding the associated eigenvalue, λ_3 ...

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 5 & 4 \\ 4 & 4 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -6 \\ -3 \\ 4 \end{pmatrix} = \lambda_3\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

$$\lambda_3 = -3$$

$$\mathbf{D} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

[4 marks]

Answer 4

(a)

$$det\left(\mathbf{A} - \lambda\mathbf{I} \right) = 0$$

$$det\begin{pmatrix} 7-\lambda & 0 & -2 \\ 0 & 5-\lambda & -2 \\ -2 & -2 & 6-\lambda \end{pmatrix} = 0$$

$$(7-\lambda)\begin{vmatrix} 5-\lambda & -2 \\ -2 & 6-\lambda \end{vmatrix} + (-2)\begin{vmatrix} 0 & 5-\lambda \\ -2 & -2 \end{vmatrix} = 0$$

$$(7-\lambda)((5-\lambda)(6-\lambda) - 4) - 2(0 + 2(5-\lambda)) = 0$$

$$\left(7-\lambda\right)\left(\lambda^2 - 11\lambda + 26\right) - 20 + 4\lambda = 0$$

$$7\lambda^2 - 77\lambda + 182 - \lambda^3 + 11\lambda^2 - 26\lambda - 20 + 4\lambda = 0$$

$$\lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0$$

As we are told that $\lambda = 9$ is a solution...

$$\begin{array}{r|l} & \lambda^2 - 9\lambda + 18 \\ \hline \lambda - 9 & \lambda^3 - 18\lambda^2 + 99\lambda - 162 \\ & \underline{\lambda^3 - 9\lambda^2} \\ & \quad -9\lambda^2 + 99\lambda \\ & \quad \underline{-9\lambda^2 + 81\lambda} \\ & \qquad 18\lambda + 162 \\ & \qquad \underline{18\lambda + 162} \\ & \qquad\qquad 0 \leftarrow \text{As expected} \end{array}$$

$$\lambda^3 - 18\lambda^2 + 99\lambda - 162 = \left(\lambda - 9\right)\left(\lambda^2 - 9\lambda + 18\right)$$

$$= (\lambda - 9)(\lambda - 3)(\lambda - 6) \Leftrightarrow \lambda_1 = 3, \lambda_2 = 6, \lambda_3 = 9$$

[5 marks]

(b) For $\lambda_1 = 3$,

$$\begin{pmatrix} 7 & 0 & -2 \\ 0 & 5 & -2 \\ -2 & -2 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \begin{cases} 7x - 2z = 3x \\ 5y - 2z = 3y \\ -2x - 2y + 6z = 3z \end{cases} \Leftrightarrow \begin{array}{r} z = 2x \\ z = y \\ 2x + 2y - 3z = 0 \end{array}$$

Let $x = 1$ giving, $z = 2$, $y = 2$: $\boldsymbol{v}_1 = k_1\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ for any constant k_1

For $\lambda_2 = 6$,

$$\begin{pmatrix} 7 & 0 & -2 \\ 0 & 5 & -2 \\ -2 & -2 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 6\begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \begin{cases} 7x - 2z = 6x \\ 5y - 2z = 6y \\ -2x - 2y + 6z = 6z \end{cases} \Leftrightarrow \begin{array}{r} x = 2z \\ y = -2z \\ y = -x \end{array}$$

Let $x = 2$ giving, $z = 1$, $y = -2$: $\boldsymbol{v}_2 = k_2\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ for any constant k_2

For $\lambda_3 = 9$,

$$\begin{pmatrix} 7 & 0 & -2 \\ 0 & 5 & -2 \\ -2 & -2 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 9\begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \left\{\begin{array}{r} 7x - 2z = 9x \\ 5y - 2z = 9y \\ -2x - 2y + 6z = 9z \end{array}\right. \Leftrightarrow \quad \begin{array}{r} z = -x \\ z = -2y \\ 2x + 2y + 3z = 0 \end{array}$$

Let $x = 2$ giving, $z = -2, \quad y = 1 \quad : \quad \boldsymbol{v}_3 = k_3\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ for any constant k_3

[5 marks]

(c) Finding all possible scalar products pairs...

$$\boldsymbol{v}_1 \bullet \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = 1 \times 2 + 2 \times (-2) + 2 \times 1 = 0$$

$$\boldsymbol{v}_1 \bullet \boldsymbol{v}_3 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = 1 \times 2 + 2 \times 1 + 2 \times (-2) = 0$$

$$\boldsymbol{v}_2 \bullet \boldsymbol{v}_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = 2 \times 2 + (-2) \times 1 + 1 \times (-2) = 0$$

Therefore, the eigenvectors found in part (b) are mutually perpendicular. ☐

[2 marks]

(d) Remembering to normalize the eigenvectors...

$$\mathbf{D} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{pmatrix} \qquad \mathbf{P} = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{pmatrix}$$

[2 marks]

Answer 5

(a)

$$\det\left(\mathbf{A} - \lambda\mathbf{I}\right) = 0$$

$$\det\begin{pmatrix} 1-\lambda & -2 & 2 \\ 0 & -3-\lambda & 4 \\ 0 & -2 & 3-\lambda \end{pmatrix} = 0$$

Expanding down the first column,

$$(1-\lambda)\begin{vmatrix} -3-\lambda & 4 \\ -2 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)((-3-\lambda)(3-\lambda)+8) = 0$$

$$\left(1-\lambda\right)\left(\lambda^2-1\right) = 0$$

$$(1-\lambda)(\lambda-1)(\lambda+1) = 0$$

The eigenvalues of **A** are 1, 1 and – 1

[2 marks]

(b) For $\lambda_1 = -1$,

$$\begin{pmatrix} 1 & -2 & 2 \\ 0 & -3 & 4 \\ 0 & -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -1\begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \begin{cases} x - 2y + 2z = -x \\ -3y + 4z = -y \\ -2y + 3z = -z \end{cases} \Leftrightarrow \quad \begin{array}{r} x - y + z = 0 \\ y = 2z \\ y = 2z \end{array}$$

Let $z = 1$ giving, $y = 2$, $x = 1$: $\boldsymbol{v}_1 = k_1\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ for any constant k_1

For $\lambda_2 = 1$,

$$\begin{pmatrix} 1 & -2 & 2 \\ 0 & -3 & 4 \\ 0 & -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1\begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \begin{cases} x - 2y + 2z = x \\ -3y + 4z = y \\ -2y + 3z = z \end{cases} \Leftrightarrow \begin{array}{r} y = z \\ y = z \\ y = z \end{array}$$

The equations do not establish a relationship involving x and so x can be assigned an arbitrary value. The simplest pair of linearly independent eigenvectors are $\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

The set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is thus a suitable basis for $\mathbb{R}^3$

[4 marks]

Answer 6

(a)

$$det\left(\mathbf{A} - \lambda\mathbf{I}\right) = 0$$

$$det\begin{pmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{pmatrix} = 0$$

Expanding along the top row,

$$(-\lambda)\begin{vmatrix} -\lambda & 0 \\ 1 & -\lambda \end{vmatrix} + 1\begin{vmatrix} 1 & -\lambda \\ 0 & 1 \end{vmatrix} = 0$$

$$-\lambda^3 + 1 = 0$$

$$\lambda_1 = 1,\ \lambda_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}\,\mathbf{i},\ \lambda_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2}\,\mathbf{i}$$

[4 marks]

(b) As **A** has distinct eigenvalues it is diagonizable over $\mathbb{C}$

[2 marks]

(c) "The long way" will work but, looking for a short cut, ...

$$\mathbf{A}^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\mathbf{A}^3 = \mathbf{A}^2\,\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}$$

Now, $2009 = 3 \times 669 + 2$ and so,

$$\begin{aligned} \mathbf{A}^{2009} &= \mathbf{A}^{3\times 669 + 2} \\ &= \left(\mathbf{A}^3\right)^{669}\ \mathbf{A}^2 \\ &= \mathbf{I}^{669}\mathbf{A}^2 \\ &= \mathbf{A}^2 \\ &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \end{aligned}$$

Matrix **A** is a permutation matrix.
A property of such a matrix is that if it is raised to a sufficient power it will equal the identity matrix.
So this short cut is not as obscure as it might at first seem !
Permutation Matrices are explored in Lecture 2 of Graph Theory I

[4 marks]

Question 7

(i) Let **A** be the general 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

The trace of any square matrix, denoted $tr\,(\mathbf{A})$, is the sum of the entries along the leading diagonal. That is, $tr\,(\mathbf{A}) = a + d$

For **A**, prove the trace is equal to the sum of the eigenvalues and that the determinant is equal to the product of the eigenvalues and also that, in consequence, its characteristic equation can be written as,

$$\lambda^2 + tr\,(\mathbf{A})\lambda + det\,(\mathbf{A}) = 0$$

[4 marks]

(ii) Find and prove a similar result for 3×3 matrices.

[6 marks]

Answer 7

(i) $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ which gives $tr\,(\mathbf{A}) = a + d$ and $det\,(\mathrm{A}) = ad - cb$

$$det\left(\mathbf{A} - \lambda\mathbf{I}\right) = 0$$

$$det\begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = 0$$

$$(a - \lambda)(d - \lambda) - cb = 0$$

$$\lambda^2 - (a + d)\lambda + ad - cb = 0$$

Now, let the roots of this equation be α and β which are, by definition, the eigenvalues of this characteristic equation.

Open University (UK) Challenge Problem from 1994, M203, Linear Algebra 4

(i) Let A be a 2×2 matrix with real eigenvalues.
Prove that A is similar to one of the following matrices;

$$()$$