1.4 Answers

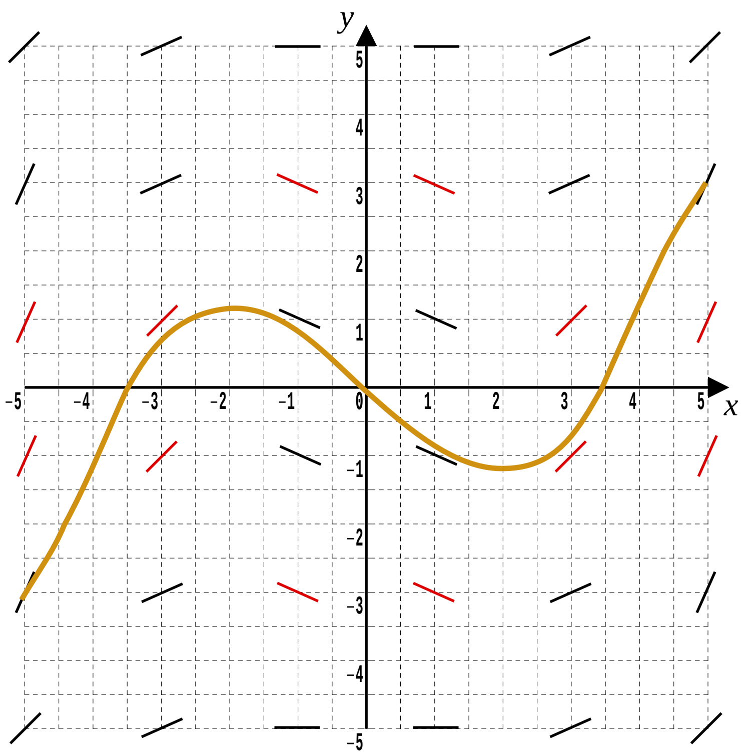
1.4.1 Solution to Teaching Example

There is mirror symmetry in both the x-axis and the y-axis so there is actually only three values to work out to complete the spreadsheet.

$$\frac{dy}{dx} = \frac{x^2 - 4}{y^2 + 4}$$

		x					
		– 5	– 3	– 1	1	3	5
y	5	0.7	0.2	– 0.1	– 0.1	0.2	0.7
	3	1.6	0.4	**– 0.2**	**– 0.2**	0.4	1.6
	1	**4.2**	**1.0**	– 0.6	– 0.6	**1.0**	**4.2**
	– 1	**4.2**	**1.0**	– 0.6	– 0.6	**1.0**	**4.2**
	– 3	1.6	0.4	**– 0.2**	**– 0.2**	0.4	1.6
	– 5	0.7	0.2	– 0.1	– 0.1	0.2	0.7

[4 marks]



[4, 2 marks]

1.4.2 Solutions to 1.3 Exercise

Answer 1

(i)

$$\frac{dy}{dx} = 0.2xy$$

$$\frac{1}{y}\frac{dy}{dx} = 0.2x \qquad \text{separate the variables}$$

Integrate both sides with respect to x

$$\int \frac{1}{y}\,dy = 0.2 \int x\ dx$$

$$\ln y = 0.1\,x^2 + c$$

$$y = e^{0.1\,x^2}\,e^c$$

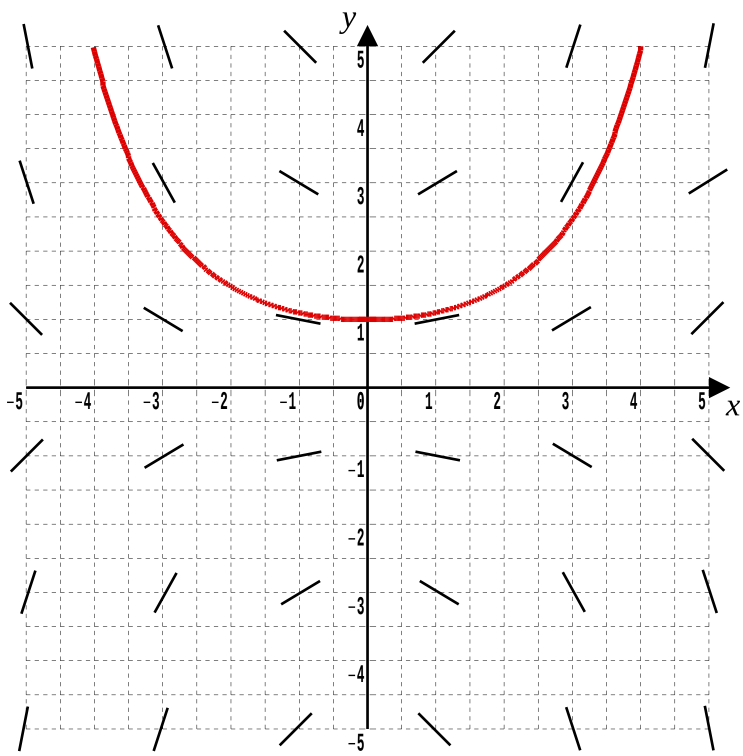
$$y = A\,e^{0.1\,x^2} \qquad \text{The General Solution}$$

Using the fact that $y = 1$ when $x = 0$ gives $c = 1$

$$\therefore\ y = e^{0.1\,x^2} \qquad \text{The Particular Solution}$$

[5 marks]

(ii)



[2 marks]

Answer 2
(i)

$$\frac{dy}{dx} = \frac{x+1}{y+1}$$

$$(y+1)\frac{dy}{dx} = x+1 \qquad \text{separate the variables}$$

Integrate both sides with respect to x

$$\int y+1\ dy = \int x+1\ dx$$

$$\frac{y^2}{2} + y = \frac{x^2}{2} + x + k \qquad \text{for some unknown constant } k$$

$$y^2 + 2y = x^2 + 2x + c \qquad \text{for some unknown constant } c$$

$$(y+1)^2 - 1 = (x+1)^2 - 1 + c \qquad \text{completing the squares}$$

$$(y+1)^2 = (x+1)^2 + c$$

Using the fact that $y = -2$ when $x = 0$ gives $c = 0$

$$y+1 = \pm(x+1)$$

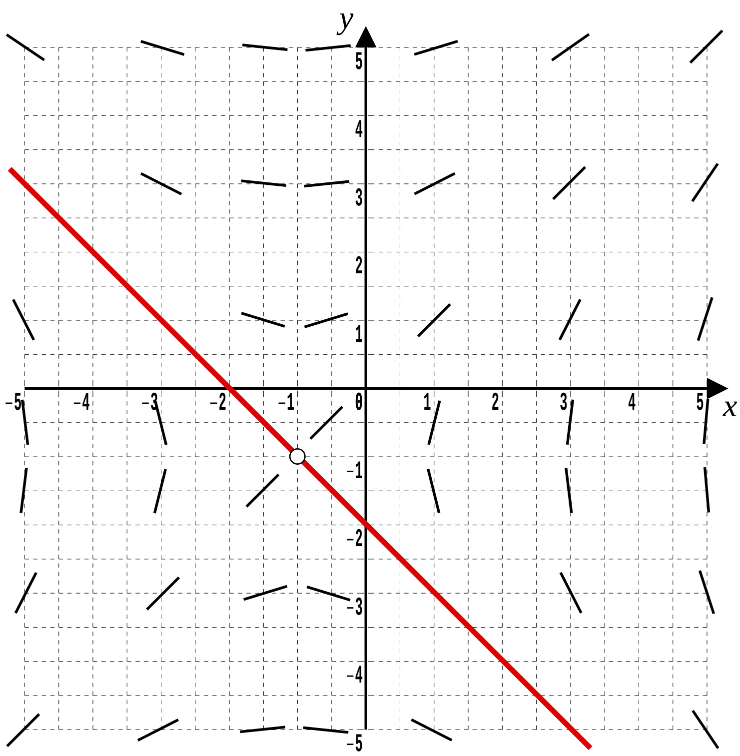
$$y = x \quad \text{or} \quad y = -x-2$$

$y = x$ is extraneous as $(0, -2)$ must have gradient of -1

$\therefore\ y = -x-2$ is the only solution

[6 marks]

(ii)



[2 marks]

Answer 3

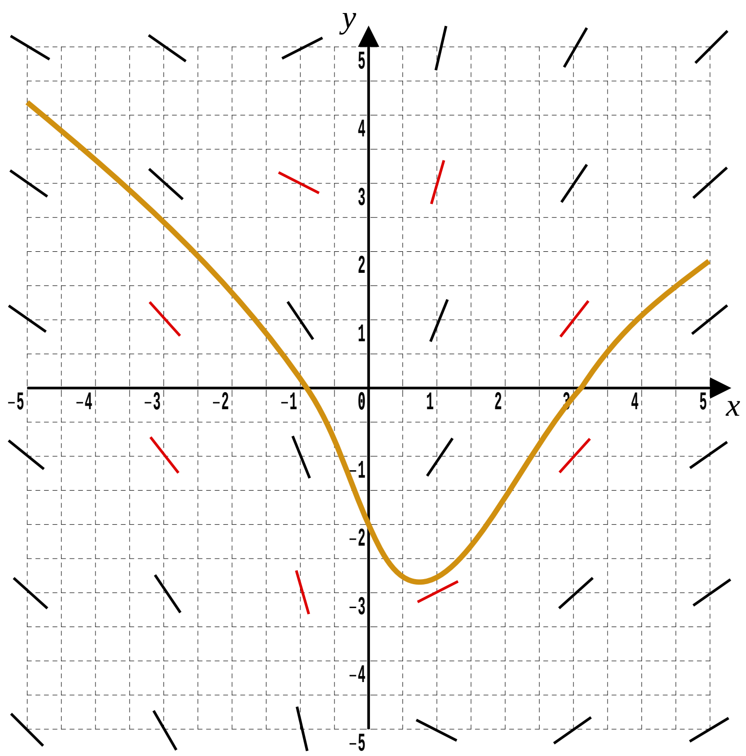
(i)

$$\frac{dy}{dx} = \frac{4x + y}{x^2 + 1} \quad \text{given that } y = -2 \text{ when } x = 0$$

		x					
		-5	-3	-1	1	3	5
y	5	-0.6	-0.7	0.5	4.5	1.7	1.0
	3	-0.7	-0.9	**-0.5**	**3.5**	1.5	0.9
	1	-0.7	**-1.1**	-1.5	2.5	**1.3**	0.8
	-1	-0.8	**-1.3**	-2.5	1.5	**1.1**	0.7
	-3	-0.9	-1.5	**-3.5**	**0.5**	0.9	0.7
	-5	-1.0	-1.7	-4.5	-0.5	0.7	0.6

[4 marks]

(ii) and **(iii)**



[4, 2 marks]

Answer 4

(i)

$$\frac{dy}{dx} = x\,e^{x-y}$$

$$= \frac{x\,e^{x}}{e^{y}}$$

$$\int e^{y}\,dy = \int x\,e^{x}\,dx$$

$$e^{y} = x\,e^{x} - \int e^{x}\,dx + k \qquad \text{Integration by parts (LIDI)}$$

$$= x\,e^{x} - e^{x} + c \qquad \text{for some unknown constants, } k \text{ and } c$$

Using the fact that $y = -3$ when $x = 0$ gives $c = \dfrac{1}{e^{3}} + 1$

$$e^{y} = e^{x}(x - 1) + \frac{1}{e^{3}} + 1$$

$$y = ln\left(e^{x}(x - 1) + \frac{1}{e^{3}} + 1 \right)$$

[4 marks]

(ii) The root are given by,

$$ln\left(e^{x}(x - 1) + \frac{1}{e^{3}} + 1 \right) = 0$$

$$e^{x}(x - 1) + \frac{1}{e^{3}} + 1 = 1$$

$$e^{x}\,(x - 1) + \frac{1}{e^{3}} = 0$$

$$-e^{x}\,(x - 1) = \frac{1}{e^{3}}$$

$$e^{x}(1 - x) = e^{-3}$$

$$ln\left(e^{x}(1 - x) \right) = ln\left(e^{-3} \right)$$

$$ln\left(e^{x} \right) + ln\left(1 - x \right) = -3$$

$$x + ln(1 - x) = -3$$

$$x + ln(1 - x) + 3 = 0$$

[4 marks]

(iii) As $ln(0)$ has no value, $(1 - x) \neq 0$ and so $x \neq 1$
The fact that the *ln* of a negative number does not exist means that
the any solution must satisfy the condition $x < 1$

[1 mark]

(iv) The Newton-Raphson formula is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

For our roots equation, $x_{n+1} = x_n - \dfrac{x_n + ln(1 - x_n) + 3}{1 - \frac{1}{1 - x_n}}$

$$x_0 = -5$$
$$x_1 = -4.750111363$$
$$x_2 = -4.749031407$$
$$x_3 = -4.749031386$$
$$x_4 = -4.749031386$$

It would seem that a root to two decimal places is $-\ 4.75$

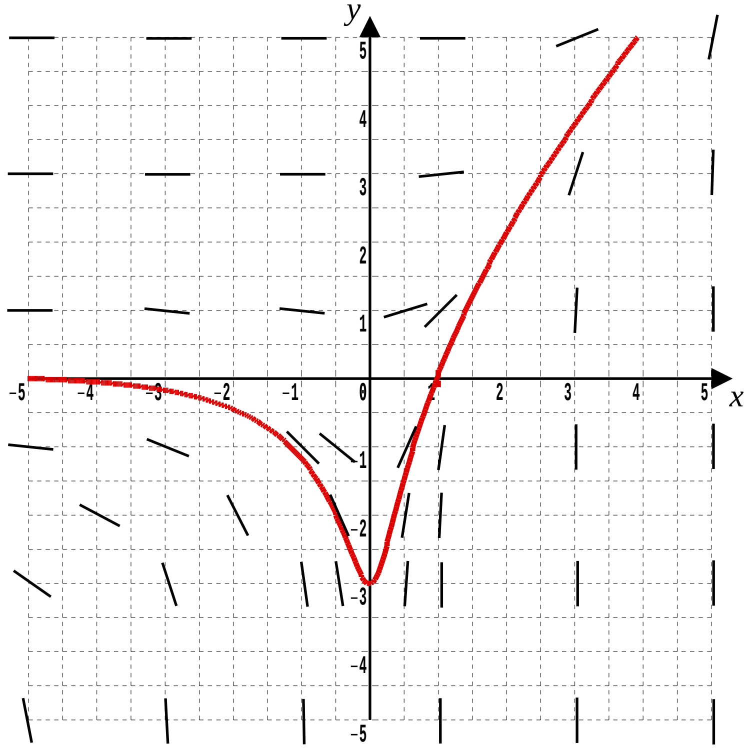
Verifying : $f(-4.745) = +0.003\,329\,911\,301$

$f(-4.755) = -0.004\,930\,957\,826$

As one answer is positive and the other negative, and the function is smooth and continuous there must be a value of x between -4.745 and -4.755 that causes $f(x) = x + ln(1 - x) + 3$ to be zero. Furthermore, all numbers in this interval round to -4.75 so that is a root to two decimal places.

[4 marks]

(v)



[2 marks]