2.7 Answers to 2.6 Exercise

Answer 1

$$\frac{dy}{dx} - 3y = e^x$$

$$P(x) = -3 \text{ and } Q(x) = e^x$$

$$\int P(x)\ dx = -3\int 1\, dx = -3x$$

(constant of integration safely omitted)

$$I = e^{\int P(x)\, dx} = e^{-3x}$$

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = e^{-3x}e^x \qquad \text{Multiplying through by } I$$

Integrate both sides with respect to x

$$\int e^{-3x}\frac{dy}{dx} - 3e^{-3x}y\ dx = \int e^{-2x}\ dx$$

$$e^{-3x}y = -\frac{1}{2}e^{-2x} + c$$

$$y = -\frac{1}{2}e^x + c\,e^{3x}$$

[5 marks]

Answer 2

$$\frac{dy}{dx} + y = x$$

$$P(x) = 1 \text{ and } Q(x) = x$$

$$\int P(x)\ dx = \int 1\, dx = x$$

(constant of integration safely omitted)

$$I = e^{\int P(x)\, dx} = e^x$$

$$e^x\frac{dy}{dx} + e^x y = xe^x \qquad \text{Multiplying through by } I$$

Integrate both sides with respect to x

$$\int e^x\frac{dy}{dx} + e^x y\ dx = \int x\,e^x\ dx$$

$$e^x y = x\,e^x - \int e^x + k \qquad \text{for some real constant, } k$$

$$= x\,e^x - e^x + c \qquad \text{for some real constant, } c$$

$$y = x - 1 + \frac{c}{e^x}$$

[5 marks]

Answer 3

(i)

$$\sqrt{x}\,\frac{dy}{dx} + y = 1$$

$$\frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{1}{\sqrt{x}} \qquad \text{OK as } x > 0$$

$$P(x) = x^{-\frac{1}{2}} \text{ and } Q(x) = x^{-\frac{1}{2}}$$

$$\int P(x)\,dx = \int x^{-\frac{1}{2}}\,dx = 2\,x^{\frac{1}{2}}$$

(constant of integration safely omitted)

$$I = e^{\int P(x)\,dx} = e^{2\sqrt{x}}$$

$$e^{2\sqrt{x}}\,\frac{dy}{dx} + \frac{e^{2\sqrt{x}}\,y}{\sqrt{x}} = \frac{e^{2\sqrt{x}}}{\sqrt{x}} \qquad \text{Multiplying through by } I$$

Integrate both sides with respect to x

$$\int e^{2\sqrt{x}}\,\frac{dy}{dx} + \frac{e^{2\sqrt{x}}\,y}{\sqrt{x}}\,dx = \int \frac{e^{2\sqrt{x}}}{\sqrt{x}}\,dx$$

$$e^{2\sqrt{x}}\,y = e^{2\sqrt{x}} + c \qquad \text{for some real constant, } c$$

$$y = 1 + \frac{c}{e^{2\sqrt{x}}}$$

[6 marks]

(ii)

$$\sqrt{x}\,\frac{dy}{dx} + y = 1$$

$$\sqrt{x}\,\frac{dy}{dx} = 1 - y$$

$$\frac{1}{1 - y}\,\frac{dy}{dx} = x^{-\frac{1}{2}}$$

$$\int \frac{1}{1 - y}\,dy = \int x^{-\frac{1}{2}}\,dx$$

$$(-1)\,ln\left|1 - y\right| = 2\,x^{\frac{1}{2}} + c_1$$

$$ln\left|1 - y\right| = -2\sqrt{x} + c_2$$

$$\left|1 - y\right| = C\,e^{-2\sqrt{x}} \qquad \text{for some positive real constant, } C$$

$$1 - y = B\,e^{-2\sqrt{x}} \qquad \text{for some real constant, } B$$

$$y = 1 + \frac{A}{e^{2\sqrt{x}}} \quad \text{as before}$$

[6 marks]

Answer 4

(i)

$$\frac{dy}{dx} + 2\,tan(x)\,y = cos^3(x)$$

$$P(x) = 2\,tan(x) \text{ and } Q(x) = cos^3(x)$$

$$\int P(x)\,dx = 2\int tan(x)\,dx = 2\,ln\left|sec(x)\right| = ln\left|sec^2(x)\right| = ln\left(sec^2(x)\right)$$

(constant of integration safely omitted)

$$I = e^{\int P(x)\,dx} = e^{ln(sec^2(x))} = sec^2(x)$$

$$\frac{dy}{dx} + 2\,tan(x)\,y = cos^3(x)$$

Multiplying through by I gives,

$$sec^2(x)\frac{dy}{dx} + 2\,tan(x)\,sec^2(x)\,y = sec^2(x)\,cos^3(x)$$

$$\int sec^2(x)\frac{dy}{dx} + 2\,tan(x)\,sec^2(x)\,y\,dx = \int cos(x)\,dx$$

$$sec^2(x)\,y = sin(x) + c$$

$$\frac{y}{cos^2(x)} = sin(x) + c$$

$$y = cos^2(x)(\,sin(x) + c\,)$$

[7 marks]

(ii) Inserting the fact that $y = 2$ at $x = 0$,

$$2 = cos^2(0)(\,sin(0) + c\,) \text{ gives } c = 2$$

$\therefore\ y = cos^2(x)(\,sin(x) + 2\,)$ is the particular solution.

[3 marks]

Answer 5

$$\frac{dy}{dx} + 2xy = x^3$$

$$P(x) = 2x \text{ and } Q(x) = x^3$$

$$\int P(x)\ dx = 2\int x\,dx = x^2$$

(constant of integration safely omitted)

$$I = e^{\int P(x)\,dx} = e^{x^2}$$

$$\frac{dy}{dx} + 2xy = x^3$$

Multiplying through by I gives,

$$e^{x^2}\frac{dy}{dx} + 2x\,e^{x^2}y = x^3\,e^{x^2}$$

$$\int e^{x^2}\frac{dy}{dx} + 2x\,e^{x^2}y = \int x^3\,e^{x^2}\,dx$$

$$e^{x^2}y = \frac{1}{2}\int x^2\left(2x\,e^{x^2}\right)dx \qquad \text{use LIDI (by parts)}$$

$$= \frac{1}{2}x^2\,e^{x^2} - \frac{1}{2}\int 2x\,e^{x^2}\,dx + k \qquad \text{for some constant, } k$$

$$= \frac{1}{2}x^2\,e^{x^2} - \frac{1}{2}e^{x^2} + c \qquad \text{for some constant, } c$$

$$y = \frac{x^2}{2} - \frac{1}{2} + c\,e^{-x^2}$$

[8 marks]