3.5 Answers to 3.4 Exercise

Answer 1

(i)

$$\frac{dy}{dx} + \frac{y}{(x+1)} = 1, \qquad x, y \in \mathbb{R},\ x \neq -1$$

$$P(x) = \frac{1}{(x+1)} \text{ and } Q(x) = \frac{1}{x^2}$$

$$\int P(x)\, dx = \int \frac{1}{(x+1)}\, dx = ln|x+1| + c$$

$$\begin{aligned} I &= e^{\int P(x)\,dx} \\ &= e^{ln|x+1|+c} \\ &= |x+1|\, e^c \\ &= B\,|x+1| \quad \text{for } B \in \mathbb{R},\ B > 0 \\ &= A(x+1) \quad \text{for } A \in \mathbb{R} \end{aligned}$$

$$\frac{dy}{dx} + \frac{y}{(x+1)} = 1, \qquad x, y \in \mathbb{R},\ x \neq -1$$

Multiplying through by I and cancelling the A gives,

$$(x+1)\frac{dy}{dx} + y = (x+1)$$

integrate both sides with respect to x

$$\int (x+1)\frac{dy}{dx} + y\ dx = \int x + 1\ dx$$

$$(x+1)\,y = \frac{x^2}{2} + x + k \qquad \text{for some constant } k$$

$$y = \frac{x^2 + 2x + c}{2(x+1)} \qquad \text{for some constant } c$$

[6 marks]

(ii) Using the condition that $y = 1$ when $x = 1$,

$$1 = \frac{1 + 2 + c}{4} \text{ gives } c = 1$$

$$\begin{aligned} y &= \frac{x^2 + 2x + 1}{2(x+1)} \\ &= \frac{(x+1)^2}{2(x+1)} \\ &= \frac{x+1}{2} \qquad x \in \mathbb{R},\ x \neq 1 \end{aligned}$$

[2 marks]

Answer 2

$$\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}, \qquad x, y \in \mathbb{R},\ x \neq 0$$

$$P(x) = \frac{1}{x} \text{ and } Q(x) = \frac{1}{x^2}$$

$$\int P(x)\ dx = \int \frac{1}{x}\, dx = ln|x| + c$$

$$\begin{aligned} I &= e^{\int P(x)\, dx} \\ &= e^{ln|x| + c} \\ &= |x|\ e^c \\ &= B|x| \text{ for } B \in \mathbb{R},\ B > 0 \\ &= Ax \qquad \text{for } A \in \mathbb{R} \end{aligned}$$

$$\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}, \qquad x, y \in \mathbb{R},\ x \neq 0$$

Multiplying through by I and cancelling the A gives,

$$x\frac{dy}{dx} + y = \frac{1}{x}$$

integrate both sides with respect to x

$$\int x\frac{dy}{dx} + y\ dx = \int \frac{1}{x}\ dx$$

$$xy = ln|x| + c$$

$$y = \frac{ln|x| + c}{x}$$

[6 marks]

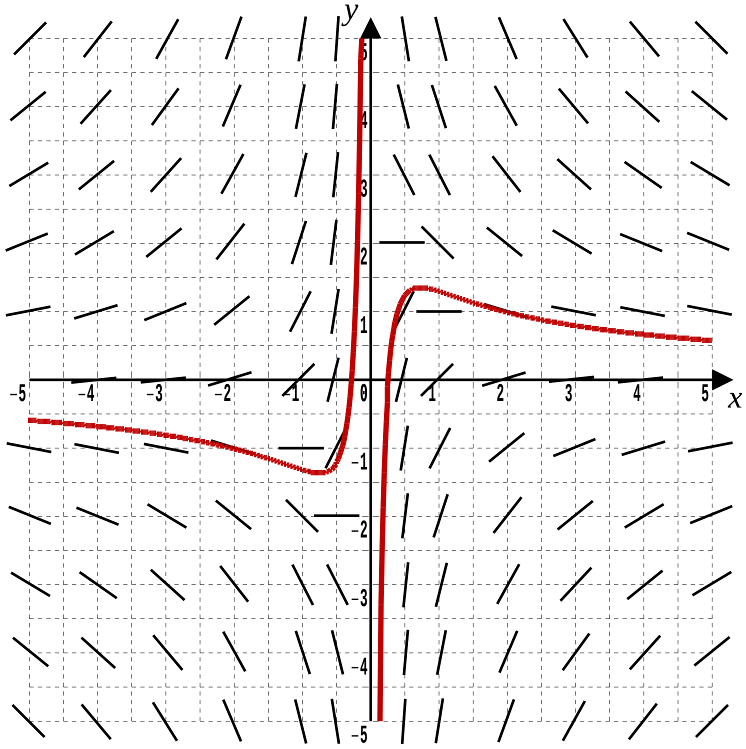
(ii) Using the condition that $y = 1$ when $x = 2$

$$1 = \frac{ln|2| + c}{2} \text{ gives } c = 2 - ln(2)$$

$$y = \frac{ln|x| + 2 - ln(2)}{x}$$

[2 marks]

(iii) The slope field plot is,



[2 marks]

Answer 3

$$tan\,(x)\ \frac{dy}{dx}\ +\ y\ =\ 3\,cos\,(2x)\ tan\,(x) \qquad\qquad 0\ <\ x\ <\ \frac{\pi}{2}$$

$$\frac{dy}{dx}\ +\ cot\,(x)\ y\ =\ 3\,cos\,(2x)$$

$$P\,(x)\ =\ cot\,(x)\ \text{ and }\ Q\,(x)\ =\ 3\,cos\,(2x)$$

$$\int P\,(x)\ dx\ =\ \int\ cot\,(x)\ dx\ =\ ln\left|\,sin\,(x)\ \right|\ +\ c$$

$$I\ =\ e^{\int P(x)\,dx}\ =\ e^{ln|\,sin(x)\,|\,+\,c}\ =\ B\left|\,sin\,(x)\ \right| \quad \text{for } B\ \in\ \mathbb{R},\ B\ >\ 0$$

$$=\ A\,sin\,(x) \qquad \text{for } A\ \in\ \mathbb{R}$$

$$\frac{dy}{dx}\ +\ cot\,(x)\ y\ =\ 3\,cos\,(2x)$$

Multiply through by I to get,

$$sin\,(x)\ \frac{dy}{dx}\ +\ \ cos\,(x)\ y\ =\ 3\,sin\,(x)\ \left(\,cos^2\,(x)\ -\ sin^2\,(x)\ \right)$$

$$=\ 3\,sin\,(x)\left(\,cos^2\,(x)\ -\ 1\ +\ cos^2\,(x)\right)$$

$$=\ 3\,sin\,(x)\left(\,2\,cos^2\,(x)\ -\ 1\ \right)$$

$$=\ 6\,sin\,(x)\ cos^2\,(x)\ -\ 3\,sin\,(x)$$

$$=\ (-\,6)\,(-\,sin\,(x)\ cos^2\,(x)\ -\ 3\,sin\,(x)$$

This has setting up $a\ \times\ f'\,(x)f\,(x)$ for a chain rule backwards

Now to integrate both sides with respect to x

$$\int sin\,(x)\ \frac{dy}{dx}\ +\ \ cos\,(x)\ y\,dx\ =\ (-\,6)\ \ \int\ \ (-\,sin\,(x))\ \ cos^2\,(x)\ dx\ -\ 3\ \int sin\,(x)\ dx$$

$$sin\,(x)\ y\ =\ (-\,6)\ \ \frac{cos^3\,(x)}{3}\ -\ 3\ \ (-\,cos\,(x))\ +\ c$$

$$y\ =\ \frac{3\,cos\,(x)\ -\ \ 2\,cos^3\,(x)\ +\ c}{sin\,(x)}$$

[6 marks]

Answer 4

(a)

$$\left(x^2 + 1\right)\frac{dy}{dx} + xy - x = 0$$

$$\frac{dy}{dx} + \frac{xy}{(x^2 + 1)} = \frac{x}{(x^2 + 1)}$$

$$P(x) = \frac{x}{(x^2 + 1)} \text{ and } Q(x) = \frac{x}{(x^2 + 1)}$$

$$\int P(x)\,dx = \frac{1}{2}\int \frac{2x}{(x^2 + 1)}\,dx = \frac{1}{2}\,ln\left(x^2 + 1\right) = ln\sqrt{x^2 + 1}$$

(constant of integration safely omitted)

$$I = e^{\int P(x)\,dx} = e^{ln\sqrt{x^2+1}} = \sqrt{x^2 + 1}$$

$$\frac{dy}{dx} + \frac{xy}{(x^2 + 1)} = \frac{x}{(x^2 + 1)}$$

Multiplying through by I gives,

$$\sqrt{x^2 + 1}\,\frac{dy}{dx} + \frac{xy}{\sqrt{x^2 + 1}} = \frac{x}{\sqrt{x^2 + 1}}$$

Integrate both sides with respect to x

$$\int \sqrt{x^2 + 1}\,\frac{dy}{dx} + \frac{xy}{\sqrt{x^2 + 1}}\,dx = \int \frac{x}{\sqrt{x^2 + 1}}\,dx$$

$$\sqrt{x^2 + 1}\,y = \sqrt{x^2 + 1} + c$$

Giving a general solution of, $\quad y = 1 + \dfrac{c}{\sqrt{x^2 + 1}}$

[6 marks]

(b) Using the constraint that $y = 2$ when $x = 3$,

$$2 = 1 + \frac{c}{\sqrt{3^2 + 1^2}}$$

$$c = \sqrt{10}$$

And so the particular solution is, $\quad y = 1 + \sqrt{\dfrac{10}{x^2 + 1}}$

[2 marks]