5.3 Answers to 5.2 Exercise

Answer 1

$$\frac{dy}{dx} = y \sinh x$$

Separate the variables

$$\frac{1}{y}\frac{dy}{dx} = \sinh x$$

Integrate both sides with respect to x

$$\int \frac{1}{y}\, dy = \int \sinh x \; dx$$

$$\ln |y| = \cosh x + c$$

$$e^{\ln|y|} = e^{\cosh x + c}$$

$$|y| = e^{\cosh x + c}$$

But the exponential function is always positive and so,

$$y = e^{\cosh x + c}$$

Using the initial conditions that $y = 1, \; x = 0$

$$1 = e^{1+c} \quad \text{gives} \quad c = -1$$

$\therefore$ The particular solution is $y = e^{\cosh x - 1}$

[4 marks]

Answer 2

(i) $M = \int e^{-3x} \sin x \; dx$

$$= e^{-3x}(-\cos x) - \int (-3)\, e^{-3x}(-\cos x)\; dx$$

$$= -e^{-3x}\cos x - 3\int e^{-3x}\cos x\, dx + c_1$$

$$= -e^{-3x}\cos x - 3\left[e^{-3x}\sin x - \int (-3)\, e^{-3x}\sin x\, dx \right] + c_2$$

$$= -e^{-3x}\cos x - 3e^{-3x}\sin x - 9\int e^{-3x}\sin x\, dx + c_2$$

$$= -e^{-3x}\cos x - 3e^{-3x}\sin x - 9M + c$$

$$10M = -e^{-3x}\cos x - 3e^{-3x}\sin x$$

$$= -e^{-3x}(\cos x + 3\sin x) + c \qquad \square$$

[4 marks]

(ii) $\dfrac{dy}{dx} - 3y = \sin x$

$$P(x) = -3 \text{ and } Q(x) = \sin x$$

$$\int P(x)\, dx = \int (-3)\, dx = -3x + c$$

(constant of integration safely omitted)

$$I = e^{\int P(x)\, dx} = e^{-3x}$$

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = e^{-3x}\sin x \qquad \text{Multiplying through by } I$$

Integrate both sides with respect to x

$$\int e^{-3x}\frac{dy}{dx} - 3e^{-3x}y \; dx = \int e^{-3x}\sin x \; dx$$

$$e^{-3x}y = M \qquad \text{(About to use part (i))}$$

$$= \frac{1}{10}\left(-e^{-3x}(\cos x + 3\sin x) + c\right)$$

$$= -\frac{e^{-3x}}{10}(\cos x + 3\sin x) + A$$

$$y = Ae^{3x} - \frac{1}{10}(\cos x + 3\sin x)$$

As $y = 0$ when $x = 0$: $0 = A \times 1 - \dfrac{1}{10}(1 + 3 \times 0)$

$$A = \frac{1}{10}$$

$$\therefore y = \frac{e^{3x}}{10} - \frac{1}{10}(\cos x + 3\sin x)$$

[4 marks]

Answer 3

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = 0$$

$$m^2 - 2m + 10 = 0 \quad \text{is the auxiliary equation}$$

$$m = \frac{2 \pm \sqrt{4 - 4 \times 1 \times 10}}{2}$$

$$= 1 \pm 3\mathrm{i}$$

$$y = e^x(A\cos(3x) + B\sin(3x))$$

for any arbitrary real constants A and B

From $y = 0$ when $x = 0$ we get,

$$0 = 1(A \times 1 + B \times 0)$$

$$A = 0$$

So we now have that,

$$y = Be^x \sin 3x$$

$$\frac{dy}{dx} = 3Be^x \cos 3x + Be^x \sin 3x$$

$$= Be^x(3\cos 3x + \sin 3x)$$

From $\frac{dy}{dx} = 3$ when $x = 0$,

$$3 = B(3)$$

$$B = 1$$

$\therefore$ The particular solution is, $y = e^x \sin 3x$

[8 marks]

Answer 4

$$\frac{d^2y}{dx^2} + k^2 y = 0$$

$$m^2 + k^2 = 0 \quad \text{is the auxiliary equation}$$

$$m = \frac{0 \pm \sqrt{0 - 4 \times 1 \times k^2}}{2}$$

$$= \frac{0 \pm \sqrt{(-1)(2k)^2}}{2}$$

$$= 0 \pm k\mathrm{i}$$

$$y = e^{0x}(A\,cos\,(kx) + B\,sin\,(kx))$$

$$= A\,cos\,(kx) + B\,sin\,(kx)$$

for any arbitrary real constants A and B

From $y = 1$ when $x = 0$ we get,

$$1 = A\,cos\,(0) + B\,sin\,(0)$$

$$A = 1$$

So we now have that,

$$y = cos\,(kx) + B\,sin\,(kx)$$

$$\frac{dy}{dx} = -k\,sin\,(kx) + Bk\,cos\,(kx)$$

From $\frac{dy}{dx} = 1$ when $x = 0$,

$$1 = -k\,sin\,(0) + Bk\,cos\,(0)$$

$$1 = Bk$$

$$B = \frac{1}{k}$$

$\therefore$ The particular solution is, $y = cos\,(kx) + \frac{1}{k}\,sin\,(kx)$

[8 marks]

Answer 5

(i) $$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 0$$

$m^2 + 2m + 5 = 0$ is the auxiliary equation

$$m = \frac{-2 \pm \sqrt{4 - 4 \times 1 \times 5}}{2}$$

$$= \frac{-2 \pm \sqrt{(-1)(16)}}{2}$$

$$= -1 \pm 2i$$

$$x = e^{-t}(A\cos(2t) + B\sin(2t))$$

for any arbitrary real constants A and B

[4 marks]

(ii) From $x = 1$ when $t = 0$ we get,

$$1 = e^0(A\cos(0) + B\sin(0))$$

$$A = 1$$

So we now have that,

$$x = e^{-t}(\cos(2t) + B\sin(2t))$$

$$\frac{dx}{dt} = e^{-t}((-2)\sin(2t) + 2B\cos(2t)) - e^{-t}(\cos(2t) + B\sin(2t))$$

$$= e^{-t}(-2 - B)\sin(2t) + e^{-t}(2B - 1)\cos(2t)$$

From $\frac{dx}{dt} = 1$ when $t = 0$,

$$1 = (2B - 1)$$

$$B = 1$$

$\therefore$ The particular solution is, $x = e^{-t}(\cos(2t) + \sin(2t))$

[4 marks]

(iii) $\cos(2t - \alpha) = \cos 2t\cos\alpha + \sin 2t\sin\alpha$

We have $\cos(2t) \times 1 + \sin(2t) \times 1$

So $\cos\alpha = 1$ and $\sin\alpha = 1$

$\therefore \tan\alpha = \frac{\sin\alpha}{\cos\alpha} = 1$ giving $\alpha = \frac{\pi}{4}$

Also, $R = \sqrt{1^2 + 1^2} = \sqrt{2}$

$$x = e^{-t}(\cos(2t) + \sin(2t))$$

$$= \sqrt{2}\,e^{-t}\cos\left(2t - \frac{\pi}{4}\right), \qquad k = -1, R = \sqrt{2}, \alpha = \frac{\pi}{4}$$

[2 marks]

(iv) The t-axis crossing points will be given by $cos\left(2t - \frac{\pi}{4}\right) = 0$

$$2t - \frac{\pi}{4} = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

$$2t = \frac{3\pi}{4}, \frac{7\pi}{4}, \ldots$$

$$t = \frac{3\pi}{8}, \frac{7\pi}{8}\ldots.$$

The x-axis crossing point is given when $t = 0$,

$$x = \sqrt{2}\, e^0 cos\left(0 - \frac{\pi}{4}\right)$$

$$= 1$$

From part (iii) you know the curve is essentially a cosine wave that has been translated $\begin{pmatrix} \frac{\pi}{4} \\ 0 \end{pmatrix}$ but its dampened by a decaying exponential envelope. Here's the accurate graph from by graph-plotter.