6.4 Answers

6.4.1 Solution to 6.2 Example

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 12y = 36x$$

Obtain the complementary function by solving the homogeneous equation,

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 12y = 0$$

$m^2 - 8m + 12 = 0$ is the auxiliary equation

$$(m - 6)(m - 2) = 0$$

Either $m = 6$ or $m = 2$

We have two distinct real roots

$\therefore$ The complementary function is $y = Ae^{2x} + Be^{6x}$

For a particular integral try $y = Ux + V$ (note the "V" is needed)

in which case $\dfrac{dy}{dx} = U$ and $\dfrac{d^2y}{dx^2} = 0$

Substitute these into the question's differential equation...

$$0 - 8U + 12(Ux + V) = 36x$$

$$-8U + 12Ux + 12V = 36x$$

Coefficients of x : $12U = 36$ gives $U = 3$

Terms independent of x : $-8U + 12V = 0$ gives $V = 2$

$\therefore$ The particular integral is $y = 3x + 2$

The general solution is $y = Ae^{2x} + Be^{6x} + 3x + 2$

[6 marks]

6.4.2 Solution to 6.3 Exercise

Answer 1

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 6$$

Obtain the complementary function by solving the homogeneous equation,

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0$$

$$m^2 - 2m - 3 = 0 \quad \text{is the auxiliary equation}$$

$$(m + 1)(m - 3) = 0$$

$$\text{Either } m = -1 \text{ or } m = 3$$

We have two distinct real roots

$\therefore$ The complementary function is $y = Ae^{-x} + Be^{3x}$

For a particular integral try $y = U$

$$\text{in which case } \frac{dy}{dx} = 0 \text{ and } \frac{d^2y}{dx^2} = 0$$

Substitute these into the question's differential equation...

$$0 - 0 - 3U = 6$$

$$U = -2$$

$\therefore$ The particular integral is $y = -2$

The general solution is $y = Ae^{-x} + Be^{3x} - 2$

[6 marks]

Answer 2

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 5e^{3x}$$

Obtain the complementary function by solving the homogeneous equation,

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

$m^2 - 3m + 2 = 0$ is the auxiliary equation

$$(m - 1)(m - 2) = 0$$

Either $m = 1$ or $m = 2$

We have two distinct real roots

$\therefore$ The complementary function is $y = Ae^{x} + Be^{2x}$

For a particular integral try $y = Ue^{3x}$

in which case $\frac{dy}{dx} = 3Ue^{3x}$ and $\frac{d^2y}{dx^2} = 9Ue^{3x}$

Substitute these into the question's differential equation...

$$9Ue^{3x} - 3 \times 3Ue^{3x} + 2Ue^{3x} = 5e^{3x}$$

$$9U - 9U + 2U = 5$$

$$U = 2.5$$

$\therefore$ The particular integral is $y = 2.5e^{3x}$

The general solution is $y = Ae^{x} + Be^{2x} + 2.5e^{3x}$

[6 marks]

Answer 3

$$\frac{d^2y}{dx^2} - y = 2x^2 - x - 3$$

Obtain the complementary function by solving the homogeneous equation,

$$\frac{d^2y}{dx^2} - y = 0$$

$m^2 - 1 = 0$ is the auxiliary equation

$(m + 1)(m - 1) = 0$

Either $m = -1$ or $m = 1$

We have two distinct real roots

$\therefore$ The complementary function is $y = Ae^{-x} + Be^{x}$

For a particular integral try $y = Ux^2 + Vx + W$

in which case $\frac{dy}{dx} = 2Ux + V$ and $\frac{d^2y}{dx^2} = 2U$

Substitute these into the question's differential equation...

$$2U - Ux^2 - Vx - W = 2x^2 - x - 3$$

Matching coefficients of x^2 : $U = -2$

Matching coefficients of x : $V = 1$

Terms independent of x : $2U - W = -3$ gives $W = -1$

$\therefore$ The particular integral is $y = -2x^2 + x - 1$

The general solution is $y = Ae^{-x} + Be^{x} - 2x^2 + x - 1$

[6 marks]

Answer 4

$$\frac{d^2y}{dx^2} + 4y = \sin x$$

Obtain the complementary function by solving the homogeneous equation,

$$\frac{d^2y}{dx^2} + 4y = 0$$

$m^2 + 4 = 0$ is the auxiliary equation

$$m^2 - (2i)^2 = 0$$

$(m + 2i)(m - 2i) = 0$ difference of two squares

Either $m = -2i$ or $m = 2i$

We have two complex roots

$\therefore$ The complementary function is $y = A\cos 2x + B\sin 2x$

For a particular integral try $y = U\cos x + V\sin x$

in which case $\frac{dy}{dx} = -U\sin x + V\cos x$

and $\frac{d^2y}{dx^2} = -U\cos x - V\sin x$

Substitute these into the question's differential equation...

$$-U\cos x - V\sin x + 4U\cos x + 4V\sin x = \sin x$$

Matching coefficients of $\cos x$: $-U + 4U = 0$ giving $U = 0$

Matching coefficients of $\sin x$: $-V + 4V = 1$ giving $V = \frac{1}{3}$

$\therefore$ The particular integral is $y = \frac{1}{3}\sin x$

The general solution is $y = A\cos 2x + B\sin 2x + \frac{1}{3}\sin x$

[6 marks]

Answer 5

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 25\cos 2x$$

Obtain the complementary function by solving the homogeneous equation,

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

$$m^2 + 2m + 1 = 0 \quad \text{is the auxiliary equation}$$

$$(m + 1)^2 = 0$$

$$m = -1$$

We have a repeated root

$\therefore$ The complementary function is $y = (A + Bx)\, e^{-x}$

For a particular integral try $y = U\cos 2x + V\sin 2x$

in which case $\dfrac{dy}{dx} = -2U\sin 2x + 2V\cos 2x$

and $\dfrac{d^2y}{dx^2} = -4U\cos 2x - 4V\sin 2x$

Substitute these into the question's differential equation...

$$-4U\cos 2x - 4V\sin 2x - 4U\sin 2x + 4V\cos 2x + U\cos 2x + V\sin 2x = 25\cos 2x$$

Matching coefficients of $\cos 2x$: $\quad -4U + 4V + U = 25$

$$-3U + 4V = 25$$

Matching coefficients of $\sin 2x$: $\quad -4V - 4U + V = 0$

$$-4U - 3V = 0$$

$$-9U + 12V = 75$$

$$ADD \quad -16U - 12V = 0$$

$$-25U = 75$$

$$U = -3 \text{ and } V = 4$$

$\therefore$ The particular integral is $y = -3\cos 2x + 4\sin 2x$

The general solution is $y = (A + Bx)\, e^{-x} - 3\cos 2x + 4\sin 2x$

[7 marks]

Answer 6

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 2\,cos\,t - sin\,t$$

Obtain the complementary function by solving the homogeneous equation,

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0$$

$m^2 + 5m + 6 = 0$ is the auxiliary equation

$$(m + 3)(m + 2) = 0$$

Either $m = -3$ or $m = -2$

We have two distinct real roots

$\therefore$ The complementary function is $x = Ae^{-3t} + Be^{-2t}$

For a particular integral try $x = U cos\,t + V sin\,t$

in which case $\dfrac{dx}{dt} = -U sin\,t + V cos\,t$

and $\dfrac{d^2x}{dt^2} = -U cos\,t - V sin\,t$

Substitute these into the question's differential equation...

$$-U cos\,t - V sin\,t - 5U\,sin\,t + 5V cos\,t + 6U\,cos\,t + 6V sin\,t = 2\,cos\,t - sin\,t$$

Matching coefficients of $cos\,t$: $-U + 5V + 6U = 2$

$$5U + 5V = 2$$

Matching coefficients of $sin\,t$: $-V - 5U + 6V = -1$

$$-5U + 5V = -1$$

$$\begin{array}{rr} & 5U + 5V = 2 \\ ADD & -5U + 5V = -1 \\ \hline & 10V = 1 \end{array}$$

$V = \dfrac{1}{10}$ and $U = \dfrac{3}{10}$

$\therefore$ The particular integral is $y = \dfrac{3}{10}cos\,t + \dfrac{1}{10}sin\,t$

The general solution is $y = Ae^{-3t} + Be^{-2t} + \dfrac{3}{10}cos\,t + \dfrac{1}{10}sin\,t$

[9 marks]