7.4 Answers

7.4.1 Solution to 7.2 Example

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} = 12$$

Obtain the complementary function by solving the homogeneous equation,

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} = 0$$

$$m^2 - 4m = 0 \quad \text{is the auxiliary equation}$$

$$m(m - 4) = 0$$

$$\text{Either } m = 0 \text{ or } m = 4$$

We have two distinct real roots

$\therefore$ The complementary function is $y = A e^{0x} + B e^{4x}$

$$y = A + B e^{4x}$$

For a particular integral we'd normally try $y = U$ but this function of a constant is already in the complementary function.

Instead, try $y = Ux$ in which case $\dfrac{dy}{dx} = U$ and $\dfrac{d^2y}{dx^2} = 0$

Substitute these into the question's differential equation...

$$0 - 4U = 12$$

$$U = -3$$

$\therefore$ The particular integral is $y = -3x$

The general solution is $y = A + B e^{4x} - 3x$

[6 marks]

7.4.2 Solutions to 7.3 Exercise

Answer 1

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = e^{-3x}$$

Obtain the complementary function by solving the homogeneous equation,

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

$$m^2 + m - 6 = 0 \quad \text{is the auxiliary equation}$$

$$(m + 3)(m - 2) = 0$$

$$\text{Either } m = -3 \text{ or } m = 2$$

We have two distinct real roots

$\therefore$ The complementary function is $y = A e^{-3x} + B e^{2x}$

For a particular integral we'd normally try $y = U e^{-3x}$ but the function $\left(\text{a multiple of } e^{-3x}\right)$ is already in the complementary function.

Instead, try $y = U x e^{-3x}$ in which case,

$$\frac{dy}{dx} = Ux(-3)e^{-3x} + Ue^{-3x}$$

$$\text{and } \frac{d^2y}{dx^2} = (-3Ux)(-3)e^{-3x} + (-3U)e^{-3x} + (-3)Ue^{-3x}$$

$$= 9Ux e^{-3x} - 6Ue^{-3x}$$

Substitute these into the question's differential equation...

$$9Ux e^{-3x} - 6Ue^{-3x} - 3Ux e^{-3x} + Ue^{-3x} - 6Ux e^{-3x} = e^{-3x}$$

$$-5U = 1$$

$$U = -\frac{1}{5}$$

$\therefore$ The particular integral is $y = -\frac{1}{5} x e^{3x}$

The general solution is $y = A e^{-3x} + B e^{2x} - \frac{1}{5} x e^{3x}$

[8 marks]

Answer 2

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} = 2x^2 - x + 1$$

(i) Obtain the complementary function by solving the homogeneous equation,

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} = 0$$

$m^2 - 6m = 0$ is the auxiliary equation

$$m(m - 6) = 0$$

Either $m = 0$ or $m = 6$

We have two distinct real roots

$\therefore$ The complementary function is $y = Ae^{0x} + Be^{6x}$

$$= A + Be^{6x}$$

[3 marks]

(ii) For a particular integral we'd normally try $y = Ux^2 + Vx + W$ but there is already a constant in the part (i) answer's complementary function.

[2 marks]

(iii) Instead, try $y = Ux^3 + Vx^2 + Wx$ in which case,

$$\frac{dy}{dx} = 3Ux^2 + 2Vx + W$$

and $\dfrac{d^2y}{dx^2} = 6Ux + 2V$

Substitute these into the question's differential equation...

$$6Ux + 2V - 18Ux^2 - 12Vx - 6W = 2x^2 - x + 1$$

$$-18Ux^2 + (6U - 12V)x + 2V - 6W = 2x^2 - x + 1$$

Matching coefficients of x^2 : $-18U = 2$ giving $U = -\dfrac{1}{9}$

Matching coefficients of x : $6U - 12V = -1$ giving $V = \dfrac{1}{36}$

Terms independent of x : $2V - 6W = 1$ gives $W = -\dfrac{17}{108}$

$\therefore$ The particular integral is $y = -\dfrac{1}{9}x^3 + \dfrac{1}{36}x^2 - \dfrac{17}{108}x$

The general solution is $y = A + Be^{6x} - \dfrac{1}{9}x^3 + \dfrac{1}{36}x^2 - \dfrac{17}{108}x$

[8 marks]

Answer 3

$$\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 36y = 6e^{-6x}$$

Obtain the complementary function by solving the homogeneous equation,

$$\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 36y = 0$$

$m^2 + 12m + 36 = 0$ is the auxiliary equation

$$(m + 6)^2 = 0$$

$$m = -6$$

We have repeated roots

$\therefore$ The complementary function is $y = (A + Bx)\, e^{-6x}$

For a particular integral we'd normally try $y = Ue^{-6x}$ but that function $\left(\text{a multiple of } e^{-6x}\right)$ is already in the complementary function.

Instead we start to consider $y = Ux\, e^{-6x}$ but this too causes a clash.

Thus we end up trying $y = Ux^2 e^{-6x}$

$$\frac{dy}{dx} = Ux^2(-6)e^{-6x} + 2Ux\, e^{-6x}$$

$$\text{and } \frac{d^2y}{dx^2} = \left(-6Ux^2\right)(-6)e^{-6x} + (-12Ux)e^{-6x}$$

$$+ (2Ux)(-6)\,Ue^{-6x} + 2Ue^{-6x}$$

$$= 36Ux^2 e^{-6x} - 24Ux\, e^{-6x} + 2Ue^{-6x}$$

Substitute these into the question's differential equation...

$$36Ux^2 e^{-6x} - 24Ux\, e^{-6x} + 2Ue^{-6x}$$

$$-72Ux^2e^{-6x} + 24Ux\, e^{-6x} + 36\,Ux^2 e^{-6x} = 6e^{-6x}$$

$$2Ue^{-6x} = 6e^{-6x}$$

$$U = 3$$

$\therefore$ The particular integral is $y = 3x^2 e^{-6x}$

The general solution is $y = \left(A + Bx + 3x^2\right)e^{-6x}$

[9 marks]

Answer 4

$$\frac{d^2y}{dx^2} + 16y = \cos 4x$$

Obtain the complementary function by solving the homogeneous equation,

$$\frac{d^2y}{dx^2} + 16y = 0$$

$$m^2 + 16 = 0 \text{ is the auxiliary equation}$$

$$m^2 - (4\mathrm{i})^2 = 0$$

$$(m + 4\mathrm{i})(m - 4\mathrm{i}) = 0 \quad \text{difference of two squares}$$

Either $m = -4\mathrm{i}$ or $m = 4\mathrm{i}$ (We have two complex roots)

$\therefore$ The complementary function is $y = A\cos 4x + B\sin 4x$

For a particular integral we'd normally try $y = U\cos 4x + V\sin 4x$ but a piece of this (a multiple of $\cos 4x$) is already in the complementary function.

Instead try, $y = Ux\cos 4x + Vx\sin 4x$

in which case $\frac{dy}{dx} = Ux(-4\sin 4x) + U\cos 4x + Vx\,4\cos 4x + V\sin 4x$

$$= (-4Ux + V)\sin 4x + (U + 4Vx)\cos 4x$$

and $\frac{d^2y}{dx^2} = (-4Ux + V)(4\cos 4x) + (-4U)\sin 4x$

$$+ (U + 4Vx)(-4\sin 4x) + (4V)\cos 4x$$

$$= (-16Ux + 8V)\cos 4x + (-8U - 16Vx)\sin 4x$$

Substitute these into the question's differential equation...

$$(-16Ux + 8V)\cos 4x + (-8U - 16Vx)\sin 4x + 16Ux\cos 4x + 16Vx\sin 4x = \cos 4x$$

$$(-16Ux + 8V + 16Ux)\cos 4x + (-8U - 16Vx + 16Vx)\sin 4x = \cos 4x$$

$$8V\cos 4x - 8U\sin 4x = \cos 4x$$

Matching coefficients of $\cos x$: $8V = 1$ giving $V = \frac{1}{8}$

Matching coefficients of $\sin x$: $8U = 0$ giving $U = 0$

$\therefore$ The particular integral is $y = \frac{1}{8}x\sin 4x$

The general solution is $y = A\cos 4x + B\sin 4x + \frac{1}{8}x\sin 4x$

[10 marks]