8.3 Answers

8.3.1 Solutions to 8.2 Exercise

Answer 1

$$y'' + 2\sqrt{2}\,y' + 2y = 10$$

Obtain the complementary function by solving the homogeneous equation,

$$y'' + 2\sqrt{2}\,y' + 2y = 0$$

$$m^2 + 2\sqrt{2}\,m + 2 = 0 \quad \text{is the auxiliary equation}$$

$$\left(m + \sqrt{2}\right)^2 = 0$$

$$m = -\sqrt{2}$$

We have a repeated root

$\therefore$ The complementary function is $y = (A + Bx)\,e^{-\sqrt{2}\,x}$

For the particular integral, $y = U$, giving $y' = y'' = 0$

Substitute these into the question's differential equation...

$$0 + 2 \times 0 + 2U = 10$$

$$U = 5$$

$\therefore$ The particular integral is $y = 5$

The general solution is $y = (A + Bx)\,e^{-\sqrt{2}\,x} + 5$

[5 marks]

Answer 2

$$y'' - y' - 12y = 144x$$

Obtain the complementary function by solving the homogeneous equation,

$$y'' - y' - 12y = 0$$

$m^2 - m - 12 = 0$ is the auxiliary equation

$$(m + 3)(m - 4) = 0$$

Either $m = -3$ or $m = 4$

We have two distinct real roots

$\therefore$ The complementary function is $y = Ae^{-3x} + Be^{4x}$

For a particular integral try $y = Ux + V$ (note the "V" is needed)

in which case $y' = U$ and $y'' = 0$

Substitute these into the question's differential equation...

$$0 - U - 12(Ux + V) = 144x$$

$$-U - 12Ux - 12V = 144x$$

Coefficients of x : $-12U = 144$ gives $U = -12$

Terms independent of x : $-U - 12V = 0$ gives $V = 1$

$\therefore$ The particular integral is $y = -12x + 1$

The general solution is $y = Ae^{-3x} + Be^{4x} - 12x + 1$

[6 marks]

Answer 3

$$y'' - 7y' + 10y = x\,e^x$$

Obtain the complementary function by solving the homogeneous equation,

$$y'' - 7y' + 10y = 0$$

$$m^2 - 7m + 10 = 0 \quad \text{is the auxiliary equation}$$

$$(m - 2)(m - 5) = 0$$

$$\text{Either } m = 2 \text{ or } m = 5$$

We have two distinct real roots

$\therefore$ The complementary function is $y = A\,e^{2x} + B\,e^{5x}$

For the particular integral, $y = (Ux + V)\,e^x$

$$y' = (Ux + V)\,e^x + Ue^x$$

$$= (Ux + U + V)e^x$$

$$y'' = (Ux + U + V)e^x + Ue^x$$

$$= (Ux + 2U + V)e^x$$

Substitute these into the question's differential equation...

$$(Ux + 2U + V)e^x - 7(Ux + U + V)e^x + 10(Ux + V)\,e^x = x\,e^x$$

$$4Ux - 5U + 4V = x$$

Matching coefficients of x : $4U = 1$ giving $U = \dfrac{1}{4}$

Terms independent of x : $-\dfrac{5}{4} + 4V = 0$ giving $V = \dfrac{5}{16}$

$\therefore$ The particular integral is $y = \left(\dfrac{x}{4} + \dfrac{5}{16}\right)e^x$

The general solution is $y = A\,e^{2x} + B\,e^{5x} + \left(\dfrac{x}{4} + \dfrac{5}{16}\right)e^x$

[10 marks]

Answer 4

(i) $y = Ue^{2x} \Rightarrow y' = 2Ue^{2x} \Rightarrow y'' = 4Ue^{2x}$

Substituting these into $y'' - 4y' + 13y = e^{2x}$ gives,

$$4Ue^{2x} - 4 \times 2Ue^{2x} + 13 \times Ue^{2x} = e^{2x}$$

$$4U - 8U + 13U = 1$$

$$9U = 1$$

$$U = \frac{1}{9}$$

[4 marks]

(ii) Obtain the complementary function by solving the homogeneous equation,

$$y'' - 4y' + 13y = 0$$

$m^2 - 4m + 13 = 0$ is the auxiliary equation

$$m = \frac{4 \pm \sqrt{16 - 4 \times 1 \times 13}}{2}$$

$$= \frac{4 \pm \sqrt{(-1)(36)}}{2}$$

$$= \frac{4 \pm 6\mathrm{i}}{2}$$

$$= 2 \pm 3\mathrm{i}$$

We have two complex roots

$\therefore$ The complementary function is $y = e^{2x}(A\cos 3x + B\sin 3x)$

The general solution is $y = e^{2x}(A\cos 3x + B\sin 3x) + \frac{1}{9}e^{2x}$

[4 marks]

Answer 5

(i) $y'' - 4y' + 4y = 4e^{2x}$

Obtain the complementary function by solving the homogeneous equation,

$$y'' - 4y' + 4y = 0$$

$$m^2 - 4m + 4 = 0 \quad \text{is the auxiliary equation}$$

$$(m - 2)^2 = 0$$

$$m = 2$$

We have a repeated root

$\therefore$ The complementary function is $y = (A + Bx)\, e^{2x}$

[3 marks]

(ii) Both Ue^{2x} and $Ux\, e^{2x}$ are part of the complementary function.

[2 marks]

(iii) For the particular integral $y = Ux^2 e^{2x}$

$$y' = Ux^2 (2) e^{2x} + 2Ux\, e^{2x}$$

$$= \left(2Ux^2 + 2Ux \right) e^{2x}$$

$$y'' = \left(2Ux^2 + 2Ux \right) 2\, e^{2x} + \left(4Ux + 2U \right) e^{2x}$$

$$= \left(4Ux^2 + 8Ux + 2U \right) e^{2x}$$

Substitute these into the question's differential equation...

$$\left(4Ux^2 + 8Ux + 2U \right) e^{2x} - 4\left(2Ux^2 + 2Ux \right) e^{2x} + 4Ux^2 e^{2x} = 4\, e^{2x}$$

$$2U = 4$$

$$U = 2$$

$\therefore$ The particular integral is $y = 2\, x^2 e^{2x}$

The general solution is $y = \left(A + Bx + 2\, x^2 \right) e^{2x}$

[6 marks]

Answer 6

(i) $y = U\sin 2x \Rightarrow y' = 2U\cos 2x \Rightarrow y'' = -4U\sin 2x$

Substituting these into $y'' + y = 3\sin 2x$ gives,

$$-4U\sin 2x + U\sin 2x = 3\sin 2x$$

$$-4U + U = 3$$

$$-3U = 3$$

$$U = -1$$

[4 marks]

(ii) Obtain the complementary function by solving the homogeneous equation,

$$y'' + y = 0$$

$$m^2 + 1 = 0 \quad \text{is the auxiliary equation}$$

$$m^2 = -1$$

$$= \pm \mathrm{i}$$

We have two complex roots

$\therefore$ The complementary function is $y = e^{0x}\,(A\cos x + B\sin x)$

The general solution is $y = (A\cos x + B\sin x) - \sin 2x$

[4 marks]