9.4 Answers

9.4.1 Solutions to 9.2 Example

$$y'' + 5y' + 4y = 4$$

Obtain the complementary function by solving the homogeneous equation,

$$y'' + 5y' + 4y = 0$$

$$m^2 + 5m + 4 = 0 \text{ is the auxiliary equation}$$

$$(m + 4)(m + 1) = 0$$

$$\text{Either } m = -4 \text{ or } m = -1$$

Two distinct roots

$\therefore$ The complementary function is $y = Ae^{-4x} + Be^{-x}$

For the particular integral, $y = U$ so $y' = y'' = 0$

Substitute these into the question's differential equation...

$$0 + 5 \times 0 + 4U = 4$$

$$U = 1$$

The particular integral is $y = 2$

The general solution is $y = Ae^{-4x} + Be^{-x} + 1$

from which $y' = -4Ae^{-4x} - Be^{-x}$

From the boundary conditions $y'(0) = 1,\ y(0) = 0$

$$A + B + 1 = 0 \text{ and } -4A - B = 1$$

Solving these simultaneously...

$$\begin{array}{lr} & -4A - B = 1 \\ \text{ADD} & A + B = -1 \\ \hline & -3A = 0 \end{array}$$

$$\text{So } A = 0 \text{ and } B = -1$$

The particular solution is,

$$y = 1 - e^{-x}$$

[8 marks]

9.4.2 Solutions to 9.3 Exercise

Answer 1

$$y'' + 7y' - 8y = 72$$

Obtain the complementary function by solving the homogeneous equation,

$$y'' + 7y' - 8y = 0$$

$$m^2 + 7m - 8 = 0 \quad \text{is the auxiliary equation}$$

$$(m + 8)(m - 1) = 0$$

$$\text{Either} \quad m = -8 \text{ or } m = 1$$

Two distinct roots

$\therefore$ The complementary function is $y = Ae^{-8x} + Be^{x}$

For the particular integral, $y = U$ so $y' = y'' = 0$

Substitute these into the question's differential equation...

$$0 + 7 \times 0 - 8U = 72$$

$$U = -9$$

The particular Integral is $y = -9$

The general solution is $y = Ae^{-8x} + Be^{x} - 9$

from which $y' = -8Ae^{-8x} + Be^{x}$

From the boundary conditions $y'(0) = 18,\ y(0) = 36$

$$-8A + B = 18 \text{ and } A + B - 9 = 36$$

Solving these simultaneously...

$$A + B = 45$$

$$\text{SUBTRACT} \quad -8A + B = 18$$

$$9A = 27$$

So $A = 3$ and $B = 42$

The particular solution is,

$$y = 3e^{-8x} + 42e^{x} - 9$$

[8 marks]

Answer 2

$$y'' + 12y' + 37y = 10e^{-4x}$$

Obtain the complementary function by solving the homogeneous equation,

$$y'' + 12y' + 37y = 0$$

$$m^2 + 12m + 37 = 0 \quad \text{is the auxiliary equation}$$

$$m = \frac{-12 \pm \sqrt{144 - 4 \times 37}}{2}$$

$$= \frac{-12 \pm \sqrt{(-1)(4)}}{2}$$

$$= -6 \pm \mathrm{i} \quad \text{(complex roots)}$$

$\therefore$ The complementary function is $y = e^{-6x}(A\cos x + B\sin x)$

Particular integral: $y = Ue^{-4x}$ so $y' = -4Ue^{-4x}$ and $y'' = 16Ue^{-4x}$

Substitute these into the question's differential equation...

$$16Ue^{-4x} - 48Ue^{-4x} + 37Ue^{-4x} = 10\ e^{-4x}$$

$$5U = 10$$

$$U = 2$$

The particular integral is $y = 2\ e^{-4x}$

The general solution is $y = e^{-6x}(A\cos x + B\sin x) + 2e^{-4x}$

from which $y' = e^{-6x}(-A\sin x + B\cos x)$

$$- 6e^{-6x}(A\cos x + B\sin x) - 8e^{-4x}$$

From the boundary conditions $y'(0) = 0,\ y(0) = 4$

$$B - 6A - 8 = 0 \text{ and } A + 2 = 4$$

So $A = 2$ and $B = 20$

The particular solution is,

$$y = e^{-6x}(2\cos x + 20\sin x) + 2e^{-4x}$$

[9 marks]

Answer 3

$$\frac{d^2y}{dx^2} + k^2 y = 0, \ y = 1 \text{ and } \frac{dy}{dx} = 1 \text{ at } x = 0$$

$$m^2 + k^2 = 0 \text{ is the auxiliary equation}$$

$$m^2 - (k\mathrm{i})^2 = 0$$

$$m^2 = (k\mathrm{i})^2$$

$$m = \pm k\mathrm{i} \qquad \text{(complex roots)}$$

$\therefore$ The complementary function is $y = A\cos kx + B\sin kx$

Which is also the general solution to the homogeneous differential equation.

$$\frac{dy}{dx} = -Ak\sin kx + Bk\cos kx$$

$$\frac{d^2y}{dx^2} = -Ak^2\cos kx - Bk^2\sin kx$$

Substitute these into the question's differential equation...

$$-Ak^2\cos kx - Bk^2\sin kx + k^2(A\cos kx + B\sin kx) = 0$$

which is consistent, as it should be

Making use of the boundary conditions,

$$y = A\cos kx + B\sin kx \Rightarrow 1 = A$$

$$\frac{dy}{dx} = -Ak\sin kx + Bk\cos kx \Rightarrow 1 = Bk \text{ so } B = \frac{1}{k}$$

$$\therefore \ y = \cos kx + \frac{1}{k}\sin kx$$

[8 marks]

Answer 4

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 27e^{-x}$$

(a) Obtain the complementary function by solving the homogeneous equation,

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 0$$

$$m^2 + 2m + 10 = 0 \quad \text{is the auxiliary equation}$$

$$m = \frac{-2 \pm \sqrt{4 - 4 \times 1 \times 10}}{2}$$

$$= \frac{-2 \pm \sqrt{(-1)(36)}}{2}$$

$$= -1 \pm 3\mathrm{i} \quad \text{(complex roots)}$$

$\therefore$ The complementary function is $y = e^{-x}(A\cos 3x + B\sin 3x)$

For a particular integral, $y = Ue^{-x}$

$$\frac{dy}{dx} = -Ue^{-x} \text{ and } \frac{d^2y}{dx^2} = Ue^{-x}$$

Substitute these into the question's differential equation...

$$Ue^{-x} - 2Ue^{-x} + 10Ue^{-x} = 27e^{-x}$$

$$9U = 27$$

$$U = 3$$

$\therefore$ The particular integral is $y = 3e^{-x}$

The general solution is $y = e^{-x}(A\cos 3x + B\sin 3x + 3)$

[6 marks]

(b) $\frac{dy}{dx} = e^{-x}(-3A\sin 3x + 3B\cos 3x) - e^{-x}(A\cos 3x + B\sin 3x + 3)$

Making use of the boundary conditions, $y = 0$ and $\frac{dy}{dx} = 0$ when $x = 0$

$$0 = A + 3 \Rightarrow A = -3$$

$$0 = 3B - A - 3 \Rightarrow B = 0$$

Particular solution: $y = 3e^{-x}(1 - \cos 3x)$

[6 marks]

Answer 5

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 26\sin 3x$$

(a) Obtain the complementary function by solving the homogeneous equation,

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} = 0$$

$$m^2 - 2m = 0 \quad \text{is the auxiliary equation}$$

$$m(m-2) = 0$$

Either $m = 0$ or $m = 2$ (two distinct real roots)

$\therefore$ The complementary function is $y = A + Be^{2x}$

For a particular integral try $y = U\cos 3x + V\sin 3x$

in which case $\dfrac{dy}{dx} = -3U\sin 3x + 3V\cos 3x$

and $\dfrac{d^2y}{dx^2} = -9U\cos 3x - 9V\sin 3x$

Substitute these into the question's differential equation...

$$-9U\cos 3x - 9V\sin 3x + 6U\sin 3x - 6V\cos 3x = 26\sin 3x$$

Matching coefficients of $\cos 3x$: $-9U - 6V = 0$

$$-3U - 2V = 0$$

Matching coefficients of $\sin 3x$: $6U - 9V = 26$

$$\begin{array}{lr} & -6U - 4V = 0 \\ ADD & 6U - 9V = 26 \\ \hline & -13V = 26 \end{array}$$

$$V = -2 \text{ and } U = \frac{4}{3}$$

$\therefore$ The particular integral is $y = \dfrac{4}{3}\cos 3x - 2\sin 3x$

The general solution is $y = A + Be^{2x} + \dfrac{4}{3}\cos 3x - 2\sin 3x$

[8 marks]

(b) $$\frac{dy}{dx} = 2Be^{2x} - 4\sin 3x - 6\cos 3x$$

Using the boundary conditions that $y = 0$ and $\frac{dy}{dx} = 0$ when $x = 0$

$$y = A + Be^{2x} + \frac{4}{3}\cos 3x - 2\sin 3x \Rightarrow 0 = A + B + \frac{4}{3}$$

$$\frac{dy}{dx} = 2Be^{2x} - 4\sin 3x - 6\cos 3x \quad \Rightarrow 0 = 2B - 6$$

So, $B = 3$ and $A = -\frac{13}{3}$

The particular solution is, $y = -\frac{13}{3} + 3e^{2x} + \frac{4}{3}\cos 3x - 2\sin 3x$

[5 marks]