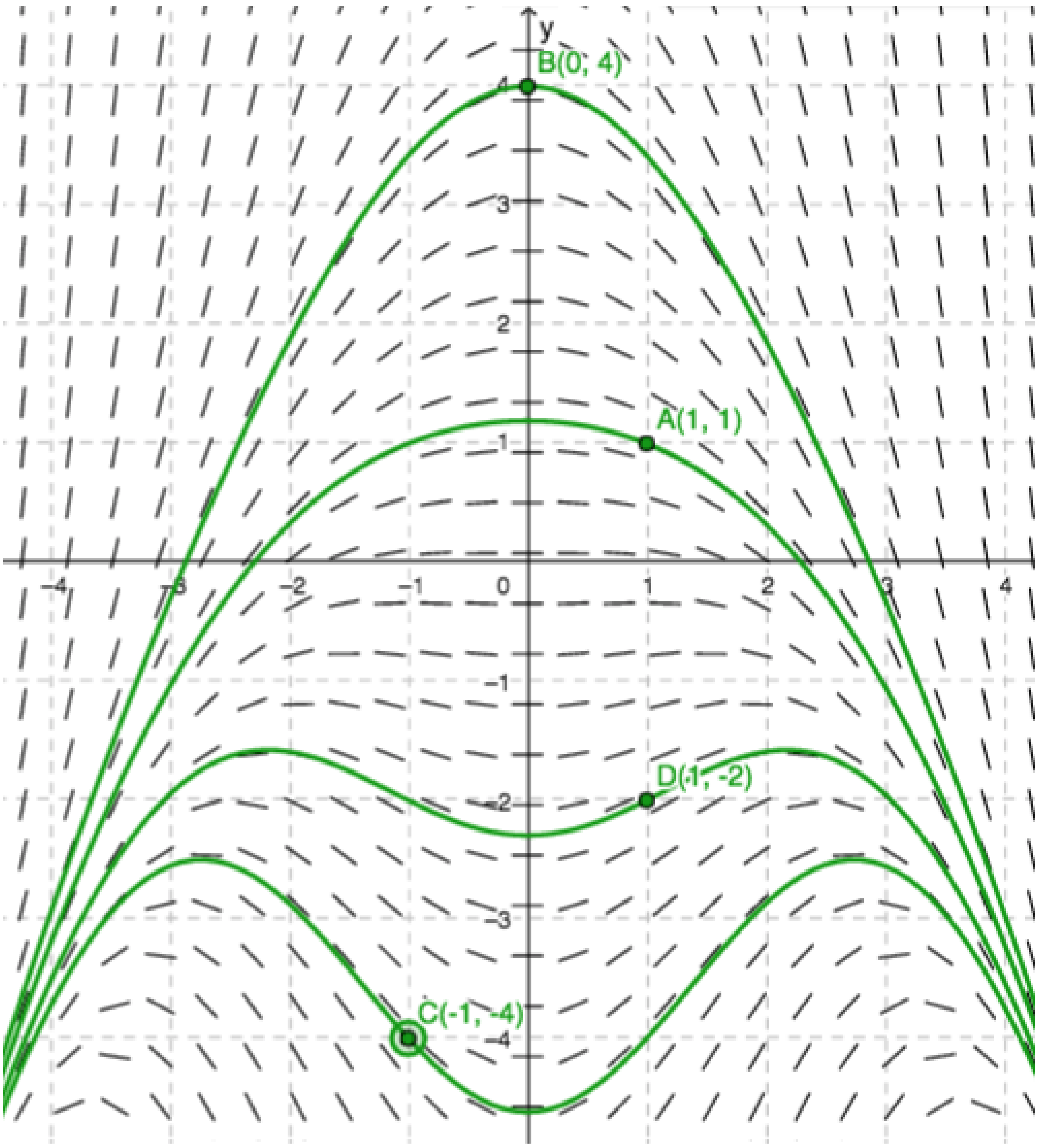
Further Pure A-Level Mathematics
Compulsory Course Component
Core 2

DIFFERENTIAL EQUATIONS II



A slope field plot for the first order linear differential equation $10\frac{dy}{dx} + 3xy + x^3 = 0$

DIFFERENTIAL EQUATIONS II

Lesson 1

Further A-Level Pure Mathematics, Core 2
Differential Equations II

1.1 Separating The Variables

Previously, in Differential Equations I, we looked at first-order differential equations that could be written in the form,

$$\frac{dy}{dx} = f(x)\,g(y)$$

These were solved by separating the variables, in other words, first rewriting the equation in the form,

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

The hope now is that both integrations can be done and the resulting algebra manipulated to give an answer in the form,

$$y = f(x)$$

As an example of the method being applied but not quite working out as planned consider solving the following differential equation given that $y = 0$, when $x = 0$,

$$\frac{dy}{dx} = \frac{x^2 - 4}{y^2 + 4}$$

This is of the required form with $f(x) = x^2 - 4$ and $g(y) = \dfrac{1}{y^2 + 4}$ and so,

we separating the variables and proceed as follows,

$$\int y^2 + 4\,dy = \int x^2 - 4\,dx$$

$$\frac{y^3}{3} + 4y = \frac{x^3}{3} - 4x + c$$

Using the information that $y = 0$ when $x = 0$, gives that $c = 0$ and so,

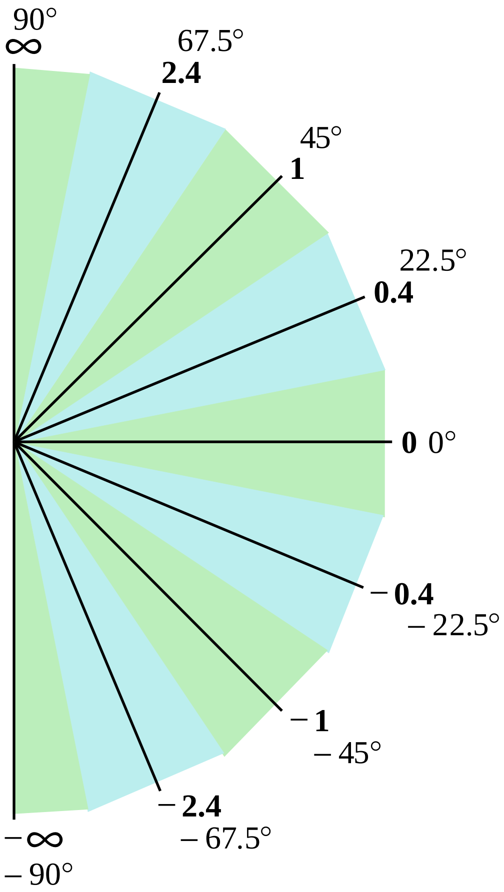
$$x^3 - y^3 = 12(x + y)$$

We've got an answer in that this equation describes the relationship between x and y. However, graphing the solution, or working out another y given an x is not straight forward because the solution is not in the form $y = f(x)$. Nor is it obvious how to get it into that form.

One way around this problem that also gives a bigger picture of the differential equation for other values of the constant c is to produce a slope field plot. This is quite a lot of work by hand; to do it repeatedly software would be used. The idea is to return to the original differential equation and insert a grid of points into it thereby finding out what the gradient is doing at each of those points.

1.2 A Slope Field Plot

When producing a slope field plot by hand, the possible gradients are reduced to eight as indicated by the following diagram. Gradient is intimately linked to the $tan(x)$ function so, for example, a slope of 22.5° corresponds to a gradient of $tan(22.5°) = 0.414$ which the diagram approximates with 0.4



Anything sufficiently close to 0.4 is drawn as a slope of 22.5°. So, for example, a gradient of 0.3 is drawn on the slope field plot as a line segment at 22.5°.

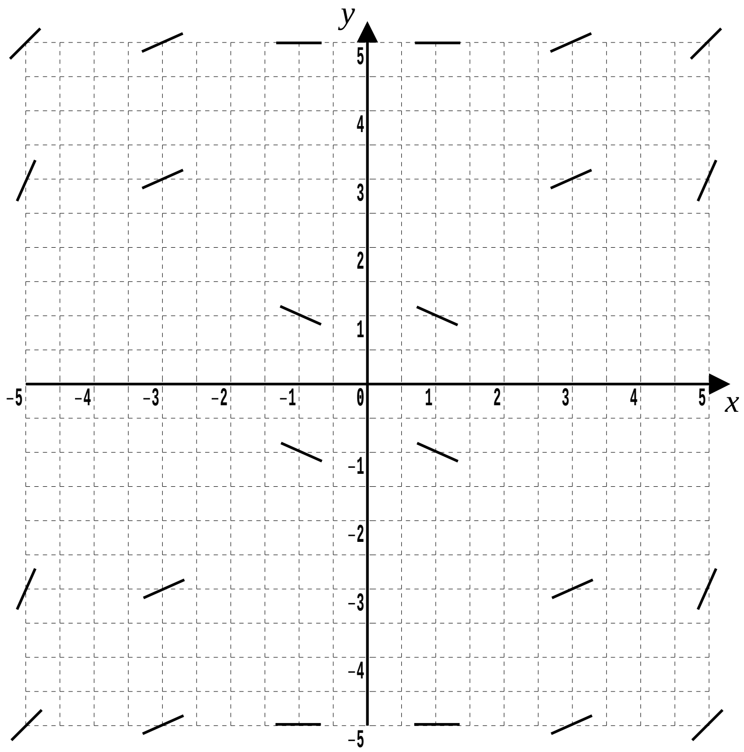
Add in the missing entries in this spreadsheet for the differential equation,

$$\frac{dy}{dx} = \frac{x^2 - 4}{y^2 + 4}$$

		x					
		− 5	− 3	− 1	1	3	5
y	5	0.7	0.2	− 0.1	− 0.1	0.2	0.7
	3	1.6	0.4			0.4	1.6
	1			− 0.6	− 0.6		
	− 1			− 0.6	− 0.6		
	− 3	1.6	0.4			0.4	1.6
	− 5	0.7	0.2	− 0.1	− 0.1	0.2	0.7

[4 marks]

From the spreadsheet, the slope field plot is drawn.
Add in the slopes corresponding to the entries you added to the spreadsheet.



[4 marks]

We are now all set to sketch the particular solution to the differential equation,

$$\frac{dy}{dx} = \frac{x^2 - 4}{y^2 + 4} \quad \text{for which } y = 0 \text{ when } x = 0$$

using only the slope field plot.

Add the solution curve to the above diagram of the slope field plot.

[2 marks]

Notice that this required no knowledge of the solution. In other words, in producing the (admittedly approximate) plot we did not need to first obtained the entwined relationship between x and y given by,

$$x^3 - y^3 = 12(x + y)$$

If you have access to a graph plotter that can plot this equation you may like to see what the exact curve looks like and so see how good your approximate answer is.

Finally, let's finish with the observation that you could draw the solution curve to the same differential equation but given a different point in the slope field. In fact the slope field plot gives an insightful overview of the possible behaviours of the differential equation.

1.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1

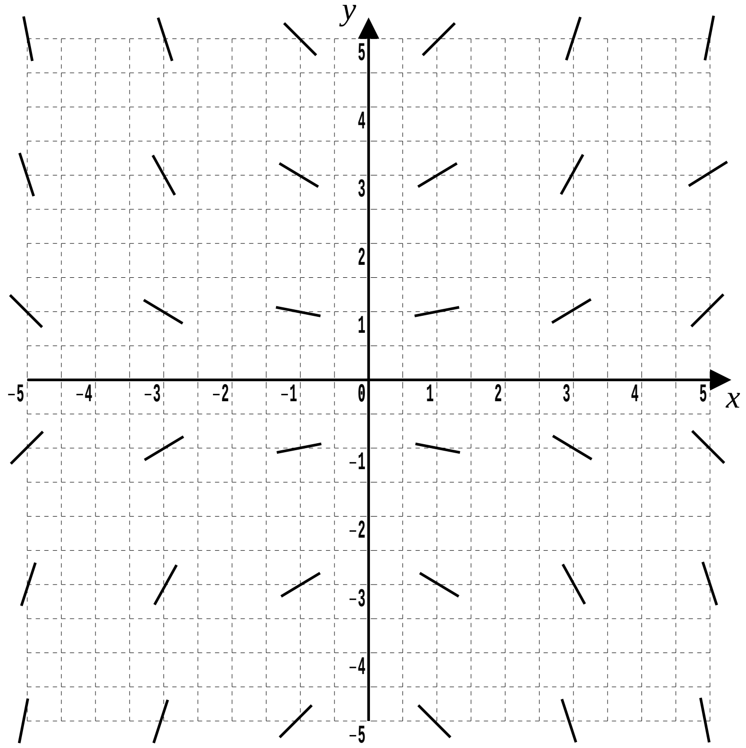
(i) By separating the variables, solve the differential equation,

$$\frac{dy}{dx} = 0.2xy \text{ given that } y = 1 \text{ when } x = 0$$

Present your solution in the form $y = A\,e^{k\,x^2}$ where A and k are constants to be found.

[5 marks]

(ii) Sketch your part (i) solution on this slope field plot.



[2 marks]

Question 2

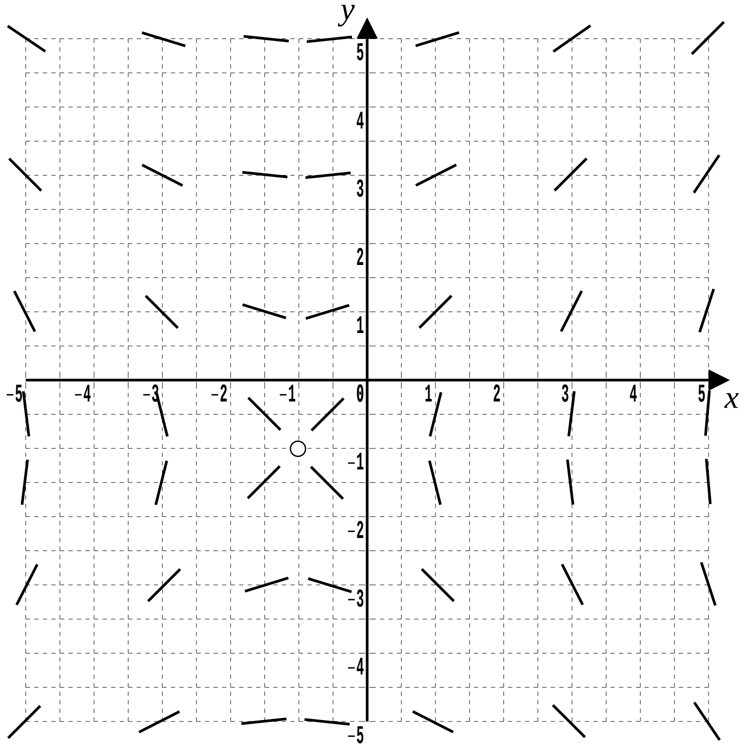
(i) By separating the variables, solve the differential equation,

$$\frac{dy}{dx} = \frac{x+1}{y+1} \text{ given that } y = -2 \text{ when } x = 0$$

presenting solution(s) in the form $y = f(x)$ the derivation of which may involve completing the square.
Check for extraneous solutions and remove any found.

[6 marks]

(ii) Here is a slope field plot for the differential equation.
Note the gradient is undefined at $(-1, -1)$.
Graph your part (i) solution(s) on this plot.



[2 marks]

Question 3

Kevin is investigating the first-order linear ordinary differential equation,

$$\frac{dy}{dx} = \frac{4x + y}{x^2 + 1} \qquad \text{given that} \quad y = -2 \quad \text{when} \quad x = 0$$

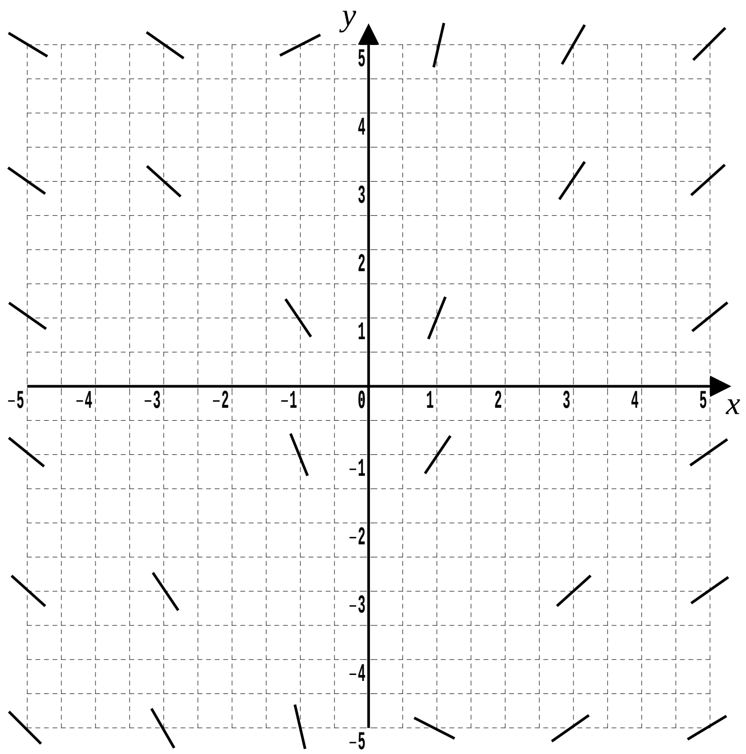
His physics teacher has advised him that this is not solvable using algebra and so he decides to produce a slope field plot to find the approximate solution curve. Here is his partly completed spreadsheet for the slope field plot,

		x					
		− 5	− 3	− 1	1	3	5
	5	− 0.6	− 0.7	0.5	4.5	1.7	1.0
	3	− 0.7	− 0.9			1.5	0.9
y	1	− 0.7		− 1.5	2.5		0.8
	− 1	− 0.8		− 2.5	1.5		0.7
	− 3	− 0.9	− 1.5			0.9	0.7
	− 5	− 1.0	− 1.7	− 4.5	− 0.5	0.7	0.6

(i) Complete Kevin's spreadsheet for him.

[4 marks]

(ii) Add in the slopes on the following slope field plot corresponding to the entries you added to Kevin's spreadsheet.



[4 marks]

(iii) Add the solution curve that passes through the point (0, – 2)

[2 marks]

Question 4

(i) By separating the variables, solve the differential equation,

$$\frac{dy}{dx} = x\,e^{x-y} \text{ given that } y = -3 \text{ when } x = 0$$

Present your answer in the form $y = f(x)$

[4 marks]

(ii) Show that the roots of your part (i) solution curve are given by,

$$x + ln(1 - x) + 3 = 0$$

[4 marks]

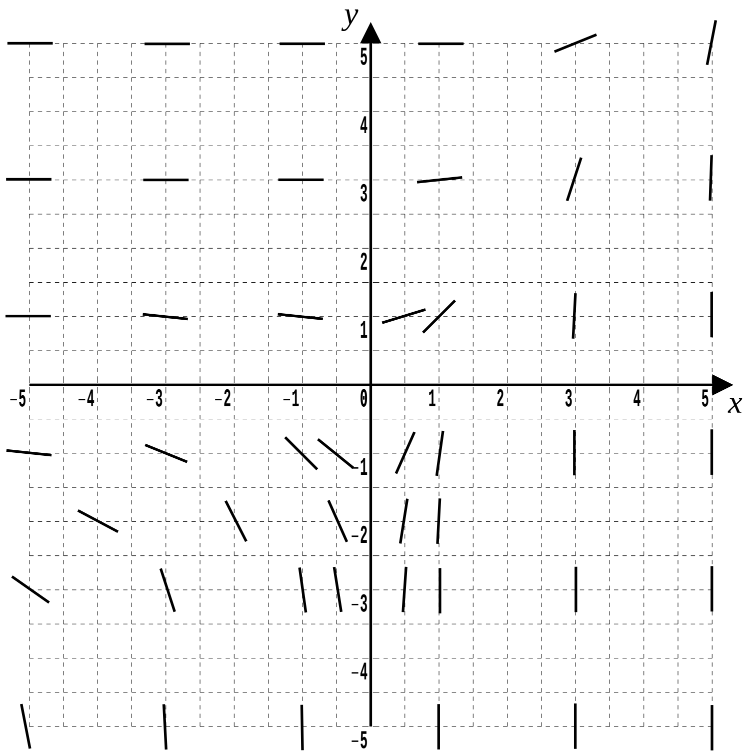
(iii) Explain why any roots of the part (i) solution curve must satisfy $x < 1$

[1 mark]

(iv) Use the Newton-Raphson starting from $x_0 = -5$ to find a root of the part (iii) equation correct to 2 decimal places.

[4 marks]

(v) Here is a slope field plot for the differential equation. Keeping in mind that the gradient is zero on the *y*-axis, graph your part (i) solution.



[2 marks]

1.4 Answers

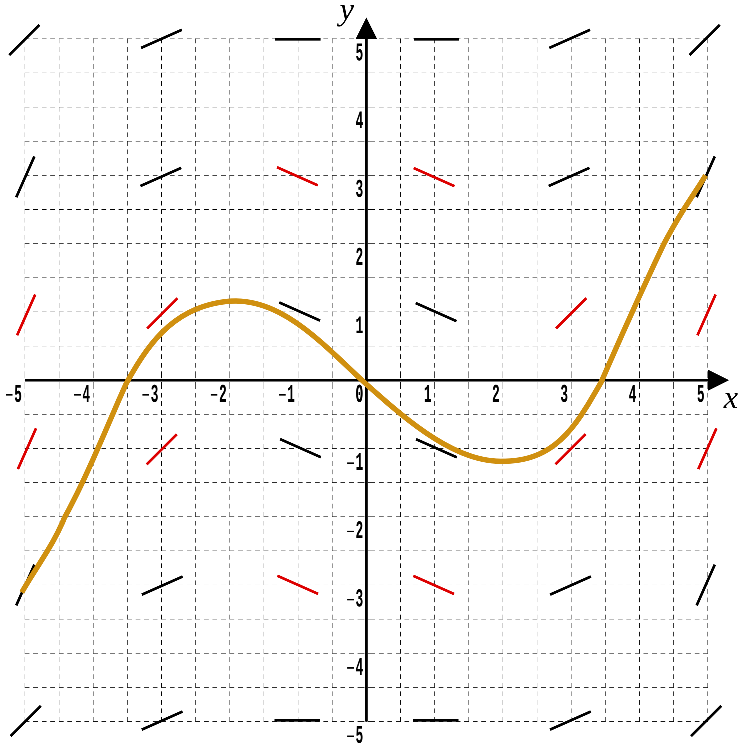
1.4.1 Solution to Teaching Example

There is mirror symmetry in both the x-axis and the y-axis so there is actually only three values to work out to complete the spreadsheet.

$$\frac{dy}{dx} = \frac{x^2 - 4}{y^2 + 4}$$

		x					
		− 5	− 3	− 1	1	3	5
y	5	0.7	0.2	− 0.1	− 0.1	0.2	0.7
	3	1.6	0.4	**− 0.2**	**− 0.2**	0.4	1.6
	1	**4.2**	**1.0**	− 0.6	− 0.6	**1.0**	**4.2**
	− 1	**4.2**	**1.0**	− 0.6	− 0.6	**1.0**	**4.2**
	− 3	1.6	0.4	**− 0.2**	**− 0.2**	0.4	1.6
	− 5	0.7	0.2	− 0.1	− 0.1	0.2	0.7

[4 marks]



[4, 2 marks]

1.4.2 Solutions to 1.3 Exercise

Answer 1

(i)

$$\frac{dy}{dx} = 0.2xy$$

$$\frac{1}{y}\frac{dy}{dx} = 0.2x \qquad \text{separate the variables}$$

Integrate both sides with respect to x

$$\int \frac{1}{y}\,dy = 0.2\int x\;dx$$

$$\ln y = 0.1\,x^2 + c$$

$$y = e^{0.1\,x^2}\,e^c$$

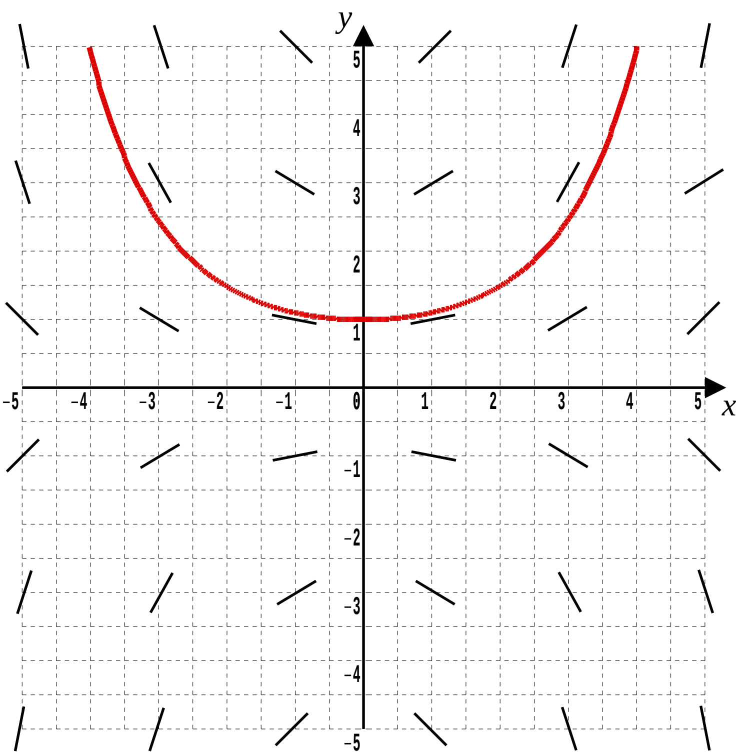
$$y = A\,e^{0.1\,x^2} \qquad \text{The General Solution}$$

Using the fact that $y = 1$ when $x = 0$ gives $c = 1$

$$\therefore\; y = e^{0.1\,x^2} \qquad \text{The Particular Solution}$$

[5 marks]

(ii)



[2 marks]

Answer 2
(i)

$$\frac{dy}{dx} = \frac{x+1}{y+1}$$

$$(y+1)\frac{dy}{dx} = x+1 \qquad \text{separate the variables}$$

Integrate both sides with respect to x

$$\int y+1 \; dy = \int x+1 \; dx$$

$$\frac{y^2}{2} + y = \frac{x^2}{2} + x + k \qquad \text{for some unknown constant } k$$

$$y^2 + 2y = x^2 + 2x + c \qquad \text{for some unknown constant } c$$

$$(y+1)^2 - 1 = (x+1)^2 - 1 + c \qquad \text{completing the squares}$$

$$(y+1)^2 = (x+1)^2 + c$$

Using the fact that $y = -2$ when $x = 0$ gives $c = 0$

$$y + 1 = \pm (x+1)$$

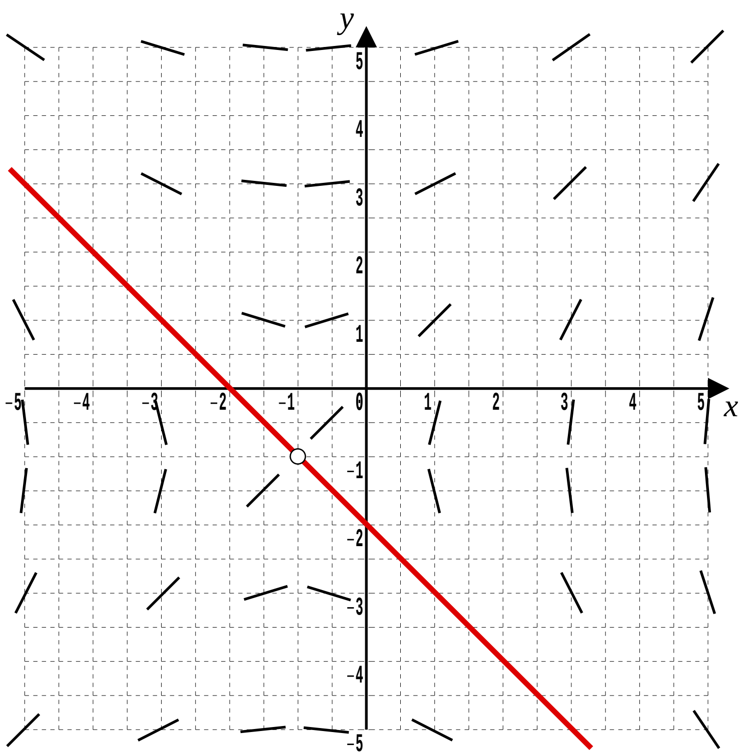
$$y = x \quad \text{or} \quad y = -x - 2$$

$y = x$ is extraneous as $(0, -2)$ must have gradient of -1

$\therefore\ y = -x - 2$ is the only solution

[6 marks]

(ii)



[2 marks]

Answer 3

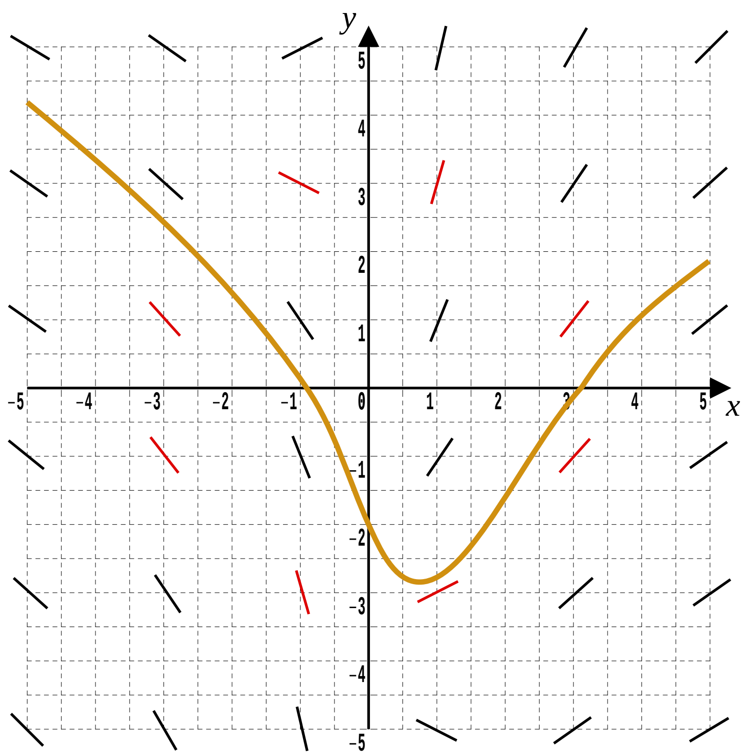
(i)

$$\frac{dy}{dx} = \frac{4x + y}{x^2 + 1} \qquad \text{given that } y = -2 \quad \text{when } x = 0$$

		x					
		– 5	– 3	– 1	1	3	5
y	5	– 0.6	– 0.7	0.5	4.5	1.7	1.0
	3	– 0.7	– 0.9	**– 0.5**	**3.5**	1.5	0.9
	1	– 0.7	**– 1.1**	– 1.5	2.5	**1.3**	0.8
	– 1	– 0.8	**– 1.3**	– 2.5	1.5	**1.1**	0.7
	– 3	– 0.9	– 1.5	**– 3.5**	**0.5**	0.9	0.7
	– 5	– 1.0	– 1.7	– 4.5	– 0.5	0.7	0.6

[4 marks]

(ii) and **(iii)**



[4, 2 marks]

Answer 4

(i)

$$\frac{dy}{dx} = x\,e^{x-y}$$

$$= \frac{x\,e^{x}}{e^{y}}$$

$$\int e^{y}\,dy = \int x\,e^{x}\,dx$$

$$e^{y} = x\,e^{x} - \int e^{x}\,dx + k \qquad \text{Integration by parts (LIDI)}$$

$$= x\,e^{x} - e^{x} + c \qquad \text{for some unknown constants, } k \text{ and } c$$

Using the fact that $y = -3$ when $x = 0$ gives $c = \frac{1}{e^{3}} + 1$

$$e^{y} = e^{x}(x - 1) + \frac{1}{e^{3}} + 1$$

$$y = ln\left(e^{x}(x - 1) + \frac{1}{e^{3}} + 1 \right)$$

[4 marks]

(ii) The root are given by,

$$ln\left(e^{x}(x - 1) + \frac{1}{e^{3}} + 1 \right) = 0$$

$$e^{x}(x - 1) + \frac{1}{e^{3}} + 1 = 1$$

$$e^{x}\,(x - 1) + \frac{1}{e^{3}} = 0$$

$$-e^{x}\,(x - 1) = \frac{1}{e^{3}}$$

$$e^{x}(1 - x) = e^{-3}$$

$$ln\left(e^{x}(1 - x) \right) = ln\left(e^{-3} \right)$$

$$ln\left(e^{x} \right) + ln\left(1 - x \right) = -3$$

$$x + ln(1 - x) = -3$$

$$x + ln(1 - x) + 3 = 0$$

[4 marks]

(iii) As $ln(0)$ has no value, $(1 - x) \neq 0$ and so $x \neq 1$
The fact that the *ln* of a negative number does not exist means that
the any solution must satisfy the condition $x < 1$

[1 mark]

(iv) The Newton-Raphson formula is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

For our roots equation, $x_{n+1} = x_n - \dfrac{x_n + ln(1 - x_n) + 3}{1 - \dfrac{1}{1 - x_n}}$

$$x_0 = -5$$
$$x_1 = -4.750111363$$
$$x_2 = -4.749031407$$
$$x_3 = -4.749031386$$
$$x_4 = -4.749031386$$

It would seem that a root to two decimal places is $-\ 4.75$

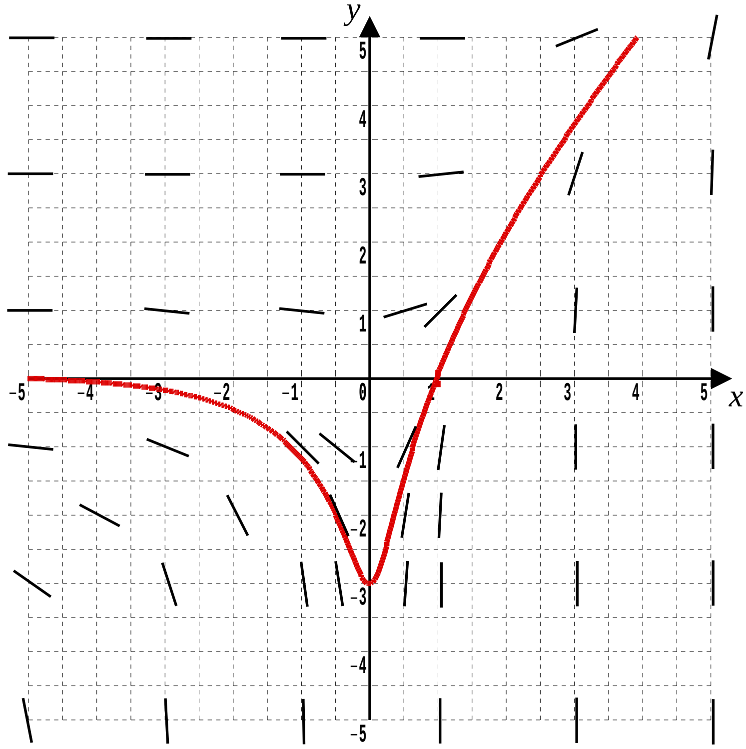
Verifying : $f(-4.745) = +0.003\,329\,911\,301$

$f(-4.755) = -0.004\,930\,957\,826$

As one answer is positive and the other negative, and the function is smooth and continuous there must be a value of x between -4.745 and -4.755 that causes $f(x) = x + ln(1 - x) + 3$ to be zero. Furthermore, all numbers in this interval round to -4.75 so that is a root to two decimal places.

[4 marks]

(v)



[2 marks]

2.1 Differential Equation Terminology

A "first order" differential equation is one in which there are only first derivatives and no derivatives of a higher order.

So, only $\frac{dy}{dx}$ in the equation and no occurrences of, for example, of $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$.

The degree, n, is the highest power of the highest derivative.

For example $\left(\frac{d^3y}{dx^3}\right)^4 + 2xy\left(\frac{d^2y}{dx^2}\right)^5 = 1$ is of order 3 and degree 4.

When a differential equation is described as "ordinary" it means that it only has a single independent variable, usually x. Typically, y then takes on a value that is determined by x, making y the dependent (upon x) variable.

In this lesson the interest is focussed upon first order, ordinary, ***linear***, differential equations. To be described as linear the the differential equation can be manipulated into the form,

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

Crucially, that y next to the $P(x)$ is just a y and NOT a function of y.

To emphasise: there can be no $\sqrt{y}$, $sin(y)$ or y^2 (for example) in the equation.

2.2 Exercise

For each of the following decide if it is a first order ordinary linear differential equation or not. For those that are not, explain why.

(i) $x^2\frac{dy}{dx} + \frac{y}{x} = sin(x)$

(ii) $y\frac{dy}{dx} + 2x = \sqrt{x^2 + 1}$

(iii) $\left(\frac{dy}{dx}\right)^2 + xy = sin(x)$

(iv) $\frac{d^2y}{dx^2} + \sqrt{x}\,y = \frac{1}{x}$

(v) $cos(x)\frac{dy}{dx} + y\,sin(x) = 1$

(vi) $\frac{dy}{dx} = y\,cosh(x)$

[6 marks]

2.3 The Integrating Factor

Differential equations that can be written in the form,

$$\frac{dy}{dx} + P(x)\, y = Q(x)$$

are interesting because, although the variables cannot be separated, there is a cunning solution technique. It involves multiplying every term by what is termed the integrating factor, *I*.

$$\text{Integrating Factor,} \quad I = e^{\int P(x)\, dx}$$

The idea is that by multiplying through by the integrating factor, the LHS becomes recognisable as being the answer to a product rule (of differentiation).

2.4 Example #1

To solve $\frac{dy}{dx} + 2xy = e^{-x^2}$ first identify that $P(x) = 2x$ and $Q(x) = e^{-x^2}$

$$\int P(x)\ dx = 2 \int x\, dx = x^2$$

If there are no occurrences of the modulus function the constant of integration may be omitted, as done here. This is because when multiplying through by the integrating factor, this constant ends up cancelling with itself. If this short-cut is not to your liking put in the constant. Beware: textbooks typically don't put it in, not even (next lesson) when a situation occurs when perhaps they should !

$$I = e^{\int P(x)\, dx} = e^{x^2}$$

$$\frac{dy}{dx} + 2xy = e^{-x^2} \qquad \text{The differential equation to be solved}$$

$$e^{x^2} \frac{dy}{dx} + 2x\, e^{x^2} y = e^{x^2} e^{-x^2} \qquad \text{Multiplying through by } I$$

Can you spot that the LHS could have come from a product rule differentiation ?

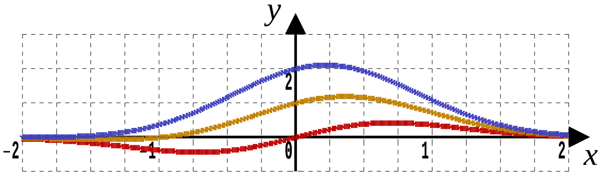
Integrate both sides with respect to *x*

$$\int e^{x^2} \frac{dy}{dx} + 2x\, e^{x^2} y\, dx = \int 1\ dx$$

$$e^{x^2} y = x + c$$

$$y = \frac{x + c}{e^{x^2}} \quad \text{is the general solution}$$

[4 marks]



Solution curves for $\frac{dy}{dx} + 2xy = e^{-x^2}$

for $c = 0$ (red), $c = 1$ (gold) and $c = 2$ (blue)

2.5 Example #2

To solve $\dfrac{dy}{dx} + y\,sin(x) = e^{cos(x)}$ noticing $P(x) = sin(x)$ and $Q(x) = e^{cos(x)}$

$$\int P(x)\ dx = \int sin(x)\ dx = -\,cos(x)$$

If there are no occurrences of the modulus function the constant of integration may be omitted, as done here.

$$I = e^{\int P(x)\,dx} = e^{-\,cos(x)}$$

$$\frac{dy}{dx} + y\,sin(x) = e^{cos(x)}$$

$$e^{-\,cos(x)}\frac{dy}{dx} + sin(x)\ e^{-\,cos(x)}y = e^{cos(x)}\,e^{-\,cos(x)}$$

integrate both sides with respect to x

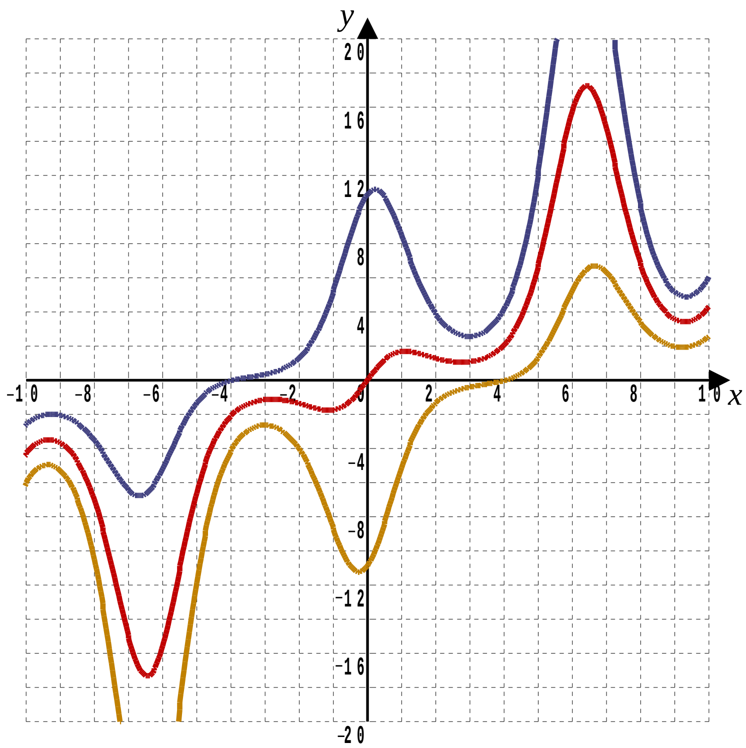
$$\int e^{-\,cos(x)}\frac{dy}{dx} + sin(x)\ e^{-\,cos(x)}y\,dx = \int 1\ dx$$

$$e^{-\,cos(x)}y = x + c \qquad \text{for any real constant } c$$

$$y = \frac{x + c}{e^{-\,cos(x)}}$$

$$y = (\,x + c\,)\,e^{cos(x)} \qquad \text{is the general solution}$$

[4 marks]



Solution curves for $\dfrac{dy}{dx} + y\,sin(x) = e^{cos(x)}$

for $c = 0$ (red), $c = -\,4$ (gold) and $c = 4$ (blue)

2.6 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1

Find the general solution of the differential equation,

$$\frac{dy}{dx} - 3y = e^{x}$$

Give an exact answer in the form $y = f(x)$

[5 marks]

Question 2

Find the general solution of the differential equation,

$$\frac{dy}{dx} + y = x$$

Give an exact answer in the form $y = f(x)$

[5 marks]

Question 3

(i) By means of an integrating factor find the general solution to,

$$\sqrt{x}\,\frac{dy}{dx} + y = 1 \qquad x, y \in \mathbb{R},\ x > 0$$

[6 marks]

(ii) This particular equation can also be solved by separating the variables. Show that doing so gets to the same answer.

[6 marks]

Question 4

(i) Find the general solution to the differential equation,

$$\frac{dy}{dx} + 2\,tan(x)\;y = cos^3(x)$$

[7 marks]

Given that $y = 2$ at $x = 0$,

(ii) Find the particular solution which satisfies this condition.

[3 marks]

Question 5

Find the general solution of the differential equation,

$$\frac{dy}{dx} + 2xy = x^3$$

Give an exact answer in the form $y = f(x)$

[8 marks]

2.7 Answers to 2.6 Exercise

Answer 1

$$\frac{dy}{dx} - 3y = e^x$$

$$P(x) = -3 \text{ and } Q(x) = e^x$$

$$\int P(x)\, dx = -3\int 1\, dx = -3x$$

(constant of integration safely omitted)

$$I = e^{\int P(x)\, dx} = e^{-3x}$$

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = e^{-3x}e^x \qquad \text{Multiplying through by } I$$

Integrate both sides with respect to x

$$\int e^{-3x}\frac{dy}{dx} - 3e^{-3x}y\ dx = \int e^{-2x}\ dx$$

$$e^{-3x}y = -\frac{1}{2}e^{-2x} + c$$

$$y = -\frac{1}{2}e^x + c\,e^{3x}$$

[5 marks]

Answer 2

$$\frac{dy}{dx} + y = x$$

$$P(x) = 1 \text{ and } Q(x) = x$$

$$\int P(x)\, dx = \int 1\, dx = x$$

(constant of integration safely omitted)

$$I = e^{\int P(x)\, dx} = e^x$$

$$e^x\frac{dy}{dx} + e^x y = xe^x \qquad \text{Multiplying through by } I$$

Integrate both sides with respect to x

$$\int e^x\frac{dy}{dx} + e^x y\ dx = \int x\,e^x\ dx$$

$$e^x y = x\,e^x - \int e^x + k \qquad \text{for some real constant, } k$$

$$= x\,e^x - e^x + c \qquad \text{for some real constant, } c$$

$$y = x - 1 + \frac{c}{e^x}$$

[5 marks]

Answer 3

(i)

$$\sqrt{x}\,\frac{dy}{dx} + y = 1$$

$$\frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{1}{\sqrt{x}} \qquad \text{OK as } x > 0$$

$$P(x) = x^{-\frac{1}{2}} \text{ and } Q(x) = x^{-\frac{1}{2}}$$

$$\int P(x)\, dx = \int x^{-\frac{1}{2}}\, dx = 2\, x^{\frac{1}{2}}$$

(constant of integration safely omitted)

$$I = e^{\int P(x)\, dx} = e^{2\sqrt{x}}$$

$$e^{2\sqrt{x}}\,\frac{dy}{dx} + \frac{e^{2\sqrt{x}}\,y}{\sqrt{x}} = \frac{e^{2\sqrt{x}}}{\sqrt{x}} \qquad \text{Multiplying through by } I$$

Integrate both sides with respect to x

$$\int e^{2\sqrt{x}}\,\frac{dy}{dx} + \frac{e^{2\sqrt{x}}\,y}{\sqrt{x}}\; dx = \int \frac{e^{2\sqrt{x}}}{\sqrt{x}}\; dx$$

$$e^{2\sqrt{x}}\,y = e^{2\sqrt{x}} + c \qquad \text{for some real constant, } c$$

$$y = 1 + \frac{c}{e^{2\sqrt{x}}}$$

[6 marks]

(ii)

$$\sqrt{x}\,\frac{dy}{dx} + y = 1$$

$$\sqrt{x}\,\frac{dy}{dx} = 1 - y$$

$$\frac{1}{1-y}\,\frac{dy}{dx} = x^{-\frac{1}{2}}$$

$$\int \frac{1}{1-y}\, dy = \int x^{-\frac{1}{2}}\, dx$$

$$(-1)\, ln\left|1 - y\right| = 2\, x^{\frac{1}{2}} + c_1$$

$$ln\left|1 - y\right| = -2\sqrt{x} + c_2$$

$$\left|1 - y\right| = C\, e^{-2\sqrt{x}} \qquad \text{for some positive real constant, } C$$

$$1 - y = B\, e^{-2\sqrt{x}} \qquad \text{for some real constant, } B$$

$$y = 1 + \frac{A}{e^{2\sqrt{x}}} \qquad \text{as before}$$

[6 marks]

Answer 4

(i)
$$\frac{dy}{dx} + 2\,tan\,(x)\;y = cos^3\,(x)$$

$$P\,(x) = 2\,tan\,(x) \;\text{ and }\; Q\,(x) = cos^3\,(x)$$

$$\int P\,(x)\;dx = 2\int tan\,(x)\;dx = 2\,ln\left|\,sec\,(x)\,\right| = ln\left|\,sec^2\,(x)\,\right| = ln\left(\,sec^2\,(x)\,\right)$$

(constant of integration safely omitted)

$$I = e^{\int P(x)\,dx} = e^{ln(\,sec^2(x)\,)} = sec^2\,(x)$$

$$\frac{dy}{dx} + 2\,tan\,(x)\;y = cos^3\,(x)$$

Multiplying through by I gives,

$$sec^2\,(x)\,\frac{dy}{dx} + 2\,tan\,(x)\;sec^2\,(x)\;y = sec^2\,(x)\;cos^3\,(x)$$

$$\int sec^2\,(x)\,\frac{dy}{dx} + 2\,tan\,(x)\;sec^2\,(x)\;y\;dx = \int cos\,(x)\;dx$$

$$sec^2\,(x)\;y = sin\,(x) + c$$

$$\frac{y}{cos^2\,(x)} = sin\,(x) + c$$

$$y = cos^2\,(x)\,(\,sin\,(x) + c\,)$$

[7 marks]

(ii) Inserting the fact that $y = 2$ at $x = 0$,

$$2 = cos^2\,(0)\,(\,sin\,(0) + c\,) \;\text{ gives }\; c = 2$$

$\therefore\; y = cos^2\,(x)\,(\,sin\,(x) + 2\,)$ is the particular solution.

[3 marks]

Answer 5

$$\frac{dy}{dx} + 2xy = x^3$$

$$P(x) = 2x \text{ and } Q(x) = x^3$$

$$\int P(x)\ dx = 2\int x\,dx = x^2$$

(constant of integration safely omitted)

$$I = e^{\int P(x)\,dx} = e^{x^2}$$

$$\frac{dy}{dx} + 2xy = x^3$$

Multiplying through by I gives,

$$e^{x^2}\frac{dy}{dx} + 2x\,e^{x^2}y = x^3 e^{x^2}$$

$$\int e^{x^2}\frac{dy}{dx} + 2x\,e^{x^2}y = \int x^3 e^{x^2}\,dx$$

$$e^{x^2}y = \frac{1}{2}\int x^2\left(2x\,e^{x^2}\right)dx \qquad \text{use LIDI (by parts)}$$

$$= \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}\int 2x\,e^{x^2}\,dx + k \qquad \text{for some constant, } k$$

$$= \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}e^{x^2} + c \qquad \text{for some constant, } c$$

$$y = \frac{x^2}{2} - \frac{1}{2} + c\,e^{-x^2}$$

[8 marks]

3.1 Integrating Factor Modulus Questions

When the method of solving a differential equation involves an integrating factor, the modulus function is frequently involved. Quite often (even in textbooks) there is a fudge along the following lines,

$$\int P(x)\ dx \ = \ 3\int \frac{1}{x}\ dx \ = \ 3\,ln(x)$$

where the modulus signs are mysteriously lost along with the constant of integration. Unfortunately the sloppiness is rewarded; it gets the correct answer ! In these lessons, however, we will include the proper logical steps which hinge upon realising (for example) that $|y| = Bx^2$ where B is a real positive number implies that $y = Ax^2$ where A is a real number.

3.2 Example #1

To solve $\frac{dy}{dx} + \frac{3y}{x} = \frac{e^x}{x^3}$ first identify that $P(x) = \frac{3}{x}$ and $Q(x) = \frac{e^x}{x^3}$

$$\int P(x)\ dx \ = \ 3\int \frac{1}{x}\ dx \ = \ 3\,ln|x| + c \ = \ ln|x^3| + c$$

$$\begin{aligned} I &= e^{\int P(x)\,dx} \\ &= e^{ln|x^3|+c} \\ &= |x^3|\ e^c \\ &= B\,|x^3| \quad \text{for } B \in \mathbb{R},\ B > 0 \\ &= Ax^3 \qquad \text{for } A \in \mathbb{R} \end{aligned}$$

As both sides of the differential equation are about to be multiplied by I there is no need for the constant of integration, the "A". If it were included then it would be immediately cancelled from both sides.

$$\frac{dy}{dx} + \frac{3y}{x} = \frac{e^x}{x^3}$$

$$x^3\frac{dy}{dx} + 3x^2y = e^x$$

integrate both sides with respect to x

$$\int x^3\frac{dy}{dx} + 3x^2y\ dx = \int e^x\ dx$$

$$x^3y = e^x + c$$

$$y = \frac{e^x + c}{x^3} \quad \text{is the general solution}$$

[6 marks]

3.3 Example #2

To solve $\frac{dy}{dx} + y\,cot\,(x) = csc\,(x) \quad x,\ y \in \mathbb{R},\ x \neq 180n,\ n \in \mathbb{Z}$

note that $P(x) = cot\,(x)$ and $Q(x) = csc\,(x)$

$$\int P(x)\ dx = \int cot\,(x)\ dx = ln|\ sin\,(x)\ | + c$$

$$\begin{aligned} I &= e^{\int P(x)\,dx} \\ &= e^{ln|\,sin(x)\,| + c} \\ &= B|\ sin\,(x)\ | \quad \text{for } B \in \mathbb{R},\ B > 0 \\ &= A\,sin\,(x) \qquad \text{for } A \in \mathbb{R} \end{aligned}$$

$$\frac{dy}{dx} + y\,cot\,(x) = csc\,(x)$$

Multiplying through by I and cancelling the A gives,

$$sin\,(x)\frac{dy}{dx} + y\frac{cos\,(x)}{sin\,(x)} \times sin\,(x) = \frac{1}{sin\,(x)} \times sin\,(x)$$

$$sin\,(x)\ \frac{dy}{dx} + cos\,(x)\ y = 1$$

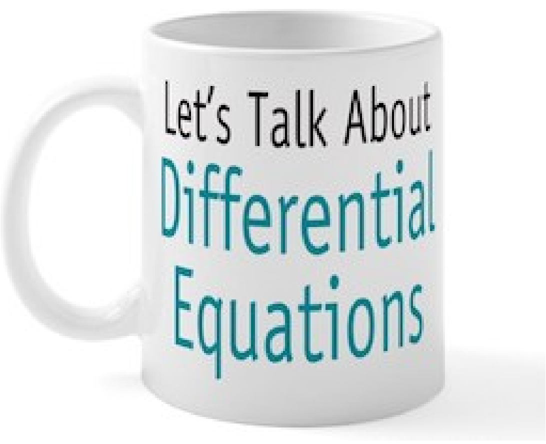
integrate both sides with respect to x

$$\int sin\,(x)\ \frac{dy}{dx} + cos\,(x)\ y\ dx = \int 1\ dx$$

$$sin\,(x)\ y = x + c \qquad \text{for any real constant } c$$

$$y = \frac{x + c}{sin\,(x)} \qquad \text{is the general solution}$$

[6 marks]



3.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 32

Question 1

(i) By means of an integrating factor, find the general solution of,

$$\frac{dy}{dx} + \frac{y}{(x+1)} = 1, \qquad x, y \in \mathbb{R}, \ x \neq -1$$

Give an exact answer in the form $y = f(x)$

[6 marks]

(ii) Find the particular solution for which $y = 1$ when $x = 1$

[2 marks]

Question 2

(i) By means of an integrating factor, find the general solution of,

$$\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}, \qquad x, y \in \mathbb{R},\ x \neq 0$$

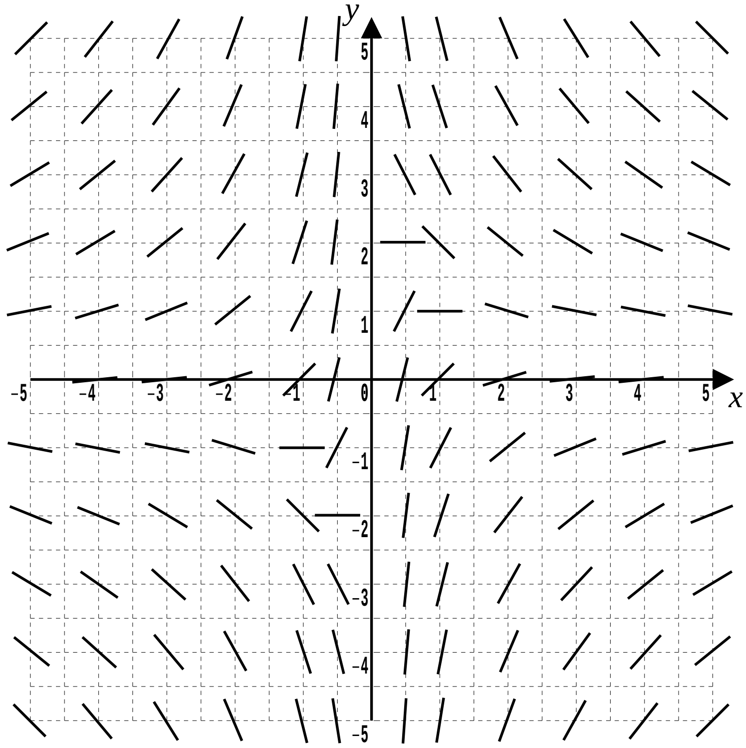
Give an exact answer in the form $y = f(x)$

[6 marks]

(ii) Find the particular solution for which $y = 1$ when $x = 2$

[2 marks]

(iii) Sketch your part (ii) solution on this slope field plot.



[2 marks]

Question 3

A-Level Examination Question from June 2015, Paper FP2, Q3 (Edexcel)

Find, in the form $y = f(x)$, the general solution of the differential equation,

$$\tan(x)\,\frac{dy}{dx} + y = 3\cos(2x)\,\tan(x), \qquad 0 < x < \frac{\pi}{2}$$

[6 marks]

Question 4

A-Level Examination Question from June 2018, Paper F2, Q2 (Edexcel)

(a) Find the general solution of the differential equation,

$$\left(x^2 + 1\right)\frac{dy}{dx} + xy - x = 0$$

giving your answer in the form $y = f(x)$

[6 marks]

(b) Find the particular solution for which $y = 2$ when $x = 3$

[2 marks]

3.5 Answers to 3.4 Exercise

Answer 1

(i)
$$\frac{dy}{dx} + \frac{y}{(x+1)} = 1, \qquad x, y \in \mathbb{R},\ x \neq -1$$

$$P(x) = \frac{1}{(x+1)} \text{ and } Q(x) = \frac{1}{x^2}$$

$$\int P(x)\, dx = \int \frac{1}{(x+1)}\, dx = ln|x+1| + c$$

$$\begin{aligned} I &= e^{\int P(x)\, dx} \\ &= e^{ln|x+1|+c} \\ &= |x+1|\, e^c \\ &= B\,|x+1| \quad \text{for } B \in \mathbb{R},\ B > 0 \\ &= A(x+1) \quad \text{for } A \in \mathbb{R} \end{aligned}$$

$$\frac{dy}{dx} + \frac{y}{(x+1)} = 1, \qquad x, y \in \mathbb{R},\ x \neq -1$$

Multiplying through by I and cancelling the A gives,

$$(x+1)\frac{dy}{dx} + y = (x+1)$$

integrate both sides with respect to x

$$\int (x+1)\frac{dy}{dx} + y\ dx = \int x+1\ dx$$

$$(x+1)\,y = \frac{x^2}{2} + x + k \qquad \text{for some constant } k$$

$$y = \frac{x^2 + 2x + c}{2(x+1)} \qquad \text{for some constant } c$$

[6 marks]

(ii) Using the condition that $y = 1$ when $x = 1$,

$$1 = \frac{1+2+c}{4} \text{ gives } c = 1$$

$$\begin{aligned} y &= \frac{x^2 + 2x + 1}{2(x+1)} \\ &= \frac{(x+1)^2}{2(x+1)} \\ &= \frac{x+1}{2} \qquad x \in \mathbb{R},\ x \neq 1 \end{aligned}$$

[2 marks]

Answer 2

$$\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}, \qquad x, y \in \mathbb{R},\ x \neq 0$$

$$P(x) = \frac{1}{x} \text{ and } Q(x) = \frac{1}{x^2}$$

$$\int P(x)\ dx = \int \frac{1}{x}\, dx = ln|x| + c$$

$$\begin{aligned} I &= e^{\int P(x)\, dx} \\ &= e^{ln|x| + c} \\ &= |x|\ e^c \\ &= B|x| \text{ for } B \in \mathbb{R},\ B > 0 \\ &= Ax \qquad \text{for } A \in \mathbb{R} \end{aligned}$$

$$\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}, \qquad x, y \in \mathbb{R},\ x \neq 0$$

Multiplying through by I and cancelling the A gives,

$$x\frac{dy}{dx} + y = \frac{1}{x}$$

integrate both sides with respect to x

$$\int x\frac{dy}{dx} + y\ dx = \int \frac{1}{x}\ dx$$

$$xy = ln|x| + c$$

$$y = \frac{ln|x| + c}{x}$$

[6 marks]

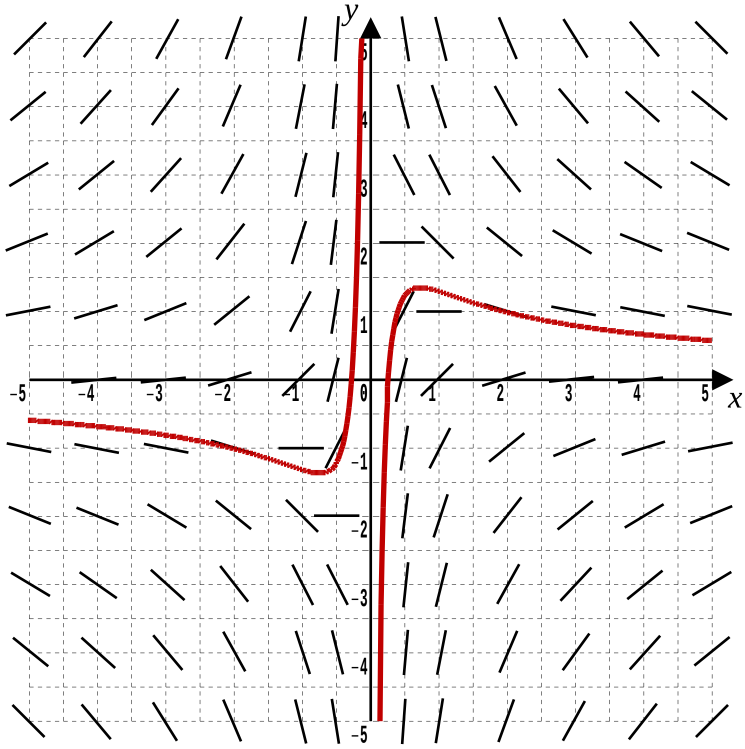
(ii) Using the condition that $y = 1$ when $x = 2$

$$1 = \frac{ln|2| + c}{2} \quad \text{gives} \quad c = 2 - ln(2)$$

$$y = \frac{ln|x| + 2 - ln(2)}{x}$$

[2 marks]

(iii) The slope field plot is,



[2 marks]

Answer 3

$$tan(x)\ \frac{dy}{dx} + y = 3\,cos(2x)\ tan(x) \qquad 0 < x < \frac{\pi}{2}$$

$$\frac{dy}{dx} + cot(x)\ y = 3\,cos(2x)$$

$$P(x) = cot(x) \ \text{ and } \ Q(x) = 3\,cos(2x)$$

$$\int P(x)\ dx = \int cot(x)\ dx = ln|\,sin(x)\,| + c$$

$$I = e^{\int P(x)\,dx} = e^{ln|\,sin(x)\,| + c} = B|\,sin(x)\,| \quad \text{for } B \in \mathbb{R},\ B > 0$$

$$= A\,sin(x) \qquad \text{for } A \in \mathbb{R}$$

$$\frac{dy}{dx} + cot(x)\ y = 3\,cos(2x)$$

Multiply through by I to get,

$$sin(x)\ \frac{dy}{dx} + cos(x)\ y = 3\,sin(x)\ \left(cos^2(x) - sin^2(x)\right)$$

$$= 3\,sin(x)\left(cos^2(x) - 1 + cos^2(x)\right)$$

$$= 3\,sin(x)\left(2\,cos^2(x) - 1\right)$$

$$= 6\,sin(x)\ cos^2(x) - 3\,sin(x)$$

$$= (-6)(-sin(x)\ cos^2(x) - 3\,sin(x)$$

This has setting up $a \times f'(x) f(x)$ for a chain rule backwards

Now to integrate both sides with respect to x

$$\int sin(x)\ \frac{dy}{dx} + cos(x)\ y\,dx = (-6)\ \int\ (-sin(x))\ cos^2(x)\ dx - 3\int sin(x)\ dx$$

$$sin(x)\ y = (-6)\ \frac{cos^3(x)}{3} - 3\ (-cos(x)) + c$$

$$y = \frac{3\,cos(x) - 2\,cos^3(x) + c}{sin(x)}$$

[6 marks]

Answer 4

(a)

$$\left(x^2 + 1 \right) \frac{dy}{dx} + xy - x = 0$$

$$\frac{dy}{dx} + \frac{xy}{(x^2 + 1)} = \frac{x}{(x^2 + 1)}$$

$$P(x) = \frac{x}{(x^2 + 1)} \text{ and } Q(x) = \frac{x}{(x^2 + 1)}$$

$$\int P(x)\, dx = \frac{1}{2} \int \frac{2x}{(x^2 + 1)} dx = \frac{1}{2} ln\left(x^2 + 1 \right) = ln \sqrt{x^2 + 1}$$

(constant of integration safely omitted)

$$I = e^{\int P(x)\, dx} = e^{ln \sqrt{x^2 + 1}} = \sqrt{x^2 + 1}$$

$$\frac{dy}{dx} + \frac{xy}{(x^2 + 1)} = \frac{x}{(x^2 + 1)}$$

Multiplying through by I gives,

$$\sqrt{x^2 + 1} \frac{dy}{dx} + \frac{xy}{\sqrt{x^2 + 1}} = \frac{x}{\sqrt{x^2 + 1}}$$

Integrate both sides with respect to x

$$\int \sqrt{x^2 + 1} \frac{dy}{dx} + \frac{xy}{\sqrt{x^2 + 1}} dx = \int \frac{x}{\sqrt{x^2 + 1}} dx$$

$$\sqrt{x^2 + 1}\, y = \sqrt{x^2 + 1} + c$$

Giving a general solution of, $\quad y = 1 + \dfrac{c}{\sqrt{x^2 + 1}}$

[6 marks]

(b) Using the constraint that $y = 2$ when $x = 3$,

$$2 = 1 + \frac{c}{\sqrt{3^2 + 1^2}}$$

$$c = \sqrt{10}$$

And so the particular solution is, $\quad y = 1 + \sqrt{\dfrac{10}{x^2 + 1}}$

[2 marks]

Lesson 4

Further A-Level Pure Mathematics, Core 2

Differential Equations II

4.1 More Involved Differential Equations

In general, a second order differential equation is one that contains second derivatives and no higher order derivatives. If it is homogeneous then each term contains the function y or one of its derivatives.

So it's of the form,

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \qquad \text{where } a, b \text{ and } c \text{ are real valued constants}$$

This differential equation is linear because there is no function applied to the y other than it being multiplied by a constant. Such equations are solved by knowing what the format of the solution is. These formats will be revealed shortly but to lessen the feeling that rabbits are being pulled out of hats, we'll first revisit the solving of a first order linear homogeneous equation.

4.2 Solving a 1st Order Linear Homogeneous Differential Equation

The differential equation to be solved is,

$$a\frac{dy}{dx} + by = 0 \qquad \text{where } a \text{ and } b \text{ are real valued constants}$$

$$\frac{dy}{dx} = -\frac{b}{a}y \qquad \text{separating the variables}$$

$$\int \frac{1}{y}\,dy = k\int 1\,dx \qquad \text{where } k = -\frac{b}{a}$$

$$ln|y| = kx + c$$

$$|y| = Be^{kx} \qquad \text{for } B > 0$$

$$y = Ae^{kx} \qquad \text{is the general solution}$$

From $k = -\dfrac{b}{a}$ we get $ak + b = 0$ which is termed the auxiliary equation

It's interestingly visually similar to where we started from; $a\dfrac{dy}{dx} + by = 0$

4.3 Solving a 2nd Order Linear Homogeneous Differential Equation

Given an equation in the form,

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \quad \text{where } a,\ b \text{ and } c \text{ are real valued constants}$$

write down the auxiliary equation,

$$am^2 + bm + c = 0$$

The two values of m that make the auxiliary equation true are performing the same roll as the value of k that made $ak + b = 0$ true previously.

Consider the discriminant, D, of the auxiliary equation.

If $\boldsymbol{D > 0}$ there are two real roots, α and β and the differential equation has a general solution,

$$y = Ae^{\alpha x} + Be^{\beta x} \quad \text{for any arbitrary real constants } A \text{ and } B$$

If $\boldsymbol{D = 0}$ there is one repeated real root α and the differential equation has a general solution,

$$y = (A + Bx)\,e^{\alpha x} \quad \text{for any arbitrary real constants } A \text{ and } B$$

If $\boldsymbol{D < 0}$ there are two complex conjugate roots, $\alpha, \beta = p \pm q\mathrm{i}$ and the differential equation has a general solution,

$$y = e^{px}(A\cos(qx) + B\sin(qx)) \quad \text{for any arbitrary real constants } A \text{ and } B$$

4.4 Three Examples (One of each type)

4.4.1 Two Real Roots

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$$

$$m^2 + 5m + 6 = 0 \quad \text{the auxiliary equation}$$

$$(m + 2)(m + 3) = 0$$

$$\text{Either } m = -2 \text{ or } m = -3$$

$$y = Ae^{-2x} + Be^{-3x} \quad \text{for any arbitrary real constants } A \text{ and } B$$

4.4.2 One (Repeated) Real Root

$$\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 25y = 0$$

$$m^2 + 10m + 25 = 0 \quad \text{the auxiliary equation}$$

$$(m + 5)(m + 5) = 0$$

$$m = -5$$

$$y = (A + Bx)\,e^{-5x} \quad \text{for any arbitrary real constants } A \text{ and } B$$

4.4.3 Two Complex Conjugate Roots

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = 0$$

$$m^2 - 4m + 13 = 0 \quad \text{the auxiliary equation}$$

$$m = \frac{4 \pm \sqrt{16 - 4 \times 1 \times 13}}{2}$$

$$= 2 \pm 3\mathrm{i}$$

$$y = e^{2x}(A\cos(3x) + B\sin(3x)) \quad \text{for any arbitrary real constants } A \text{ and } B$$

4.5 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 34

Question 1

Find the general solution to each of the following differential equations,

(i) $$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 28y = 0$$

[3 marks]

(ii) $$16\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + y = 0$$

[3 marks]

Question 2

Find the general solution to the differential equation, $\dfrac{d^2y}{dx^2} + 8\dfrac{dy}{dx} + 25y = 0$

[3 marks]

Question 3

(i) Find the general solution to the differential equation,

$$y'' + 4y' + 5y = 0$$

[3 marks]

(ii) Given that $y(0) = 0$ and $y'(0) = 2$ find the particular solution.

[5 marks]

Question 4

Find the general solution to each of the following differential equations,

(i) $15y'' - 7y' - 2y = 0$

[3 marks]

(ii) $4y'' + 20y' + 25y = 0$

[3 marks]

(iii) $y'' + \sqrt{3}\,y' + 3y = 0$

[3 marks]

Question 5

Find the solution to the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = 0$ for which $y = 0$ and $\frac{dy}{dx} = 3$ at $x = 0$

[8 marks]

4.6 Answers to 4.5 Exercise

Answer 1

(i) $$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 28y = 0$$

$m^2 - 3m - 28 = 0$ the auxiliary equation

$$(m + 4)(m - 7) = 0$$

Either $m = -4$ or $m = 7$

$y = Ae^{-4x} + Be^{7x}$ for any arbitrary real constants A and B

[3 marks]

(ii) $$16\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + y = 0$$

$16m^2 + 8m + 1 = 0$ the auxiliary equation

$$(4m + 1)(4m + 1) = 0$$

$$m = -\frac{1}{4}$$

$$y = (A + Bx)e^{-\frac{1}{4}x}$$

for any arbitrary real constants A and B

[3 marks]

Answer 2

$$\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 25y = 0$$

$m^2 + 8m + 25 = 0$ the auxiliary equation

$$m = \frac{-8 \pm \sqrt{64 - 4 \times 1 \times 25}}{2}$$

$$= -4 \pm 3i$$

$$y = e^{-4x}(A\cos(3x) + B\sin(3x))$$

for any arbitrary real constants A and B

[3 marks]

Answer 3

(i) $y'' + 4y' + 5y = 0$

$$m^2 + 4m + 5 = 0$$

$$m = \frac{-4 \pm \sqrt{16 - 4 \times 1 \times 5}}{2}$$

$$= -2 \pm \mathrm{i}$$

$$y = e^{-2x}(A\,cos(x) + B\,sin(x))$$

for any arbitrary real constants A and B

[3 marks]

(ii) As $y(0) = 0$,

$$0 = e^0(A\,cos(0) + B\,sin(0))$$

$$0 = A$$

$$\therefore \; y = e^{-2x}(B\,sin(x))$$

$$\frac{dy}{dx} = B\,e^{-2x}\,cos(x) - 2B\,e^{-2x}\,sin(x)$$

As $y'(0) = 2$,

$$2 = B\,e^0\,cos(0) - 2B\,e^0\,sin(0)$$

$$2 = B$$

The particular solution is thus,

$$y = 2\,e^{-2x}\,sin(x)$$

[5 marks]

Answer 4

(i) $15y'' - 7y' - 2y = 0$

$15m^2 - 7m - 2y = 0$ the auxiliary equation

$(5m + 1)(3m - 2) = 0$

Either $m = -\dfrac{1}{5}$ or $m = \dfrac{2}{3}$

$y = Ae^{-\frac{1}{5}x} + Be^{\frac{2}{3}x}$ for any arbitrary real constants A and B

[3 marks]

(ii) $4y'' + 20y' + 25y = 0$

$4m^2 + 20m + 25 = 0$ the auxiliary equation

$(2m + 5)(2m + 5) = 0$

$$m = -\frac{5}{2}$$

$$y = (A + Bx)\, e^{-\frac{5}{2}x}$$

for any arbitrary real constants A and B

[3 marks]

(iii) $y'' + \sqrt{3}\, y' + 3y = 0$

$m^2 + \sqrt{3}\, m + 3 = 0$ the auxiliary equation

$$m = \frac{-\sqrt{3} \pm \sqrt{3 - 4 \times 1 \times 3}}{2}$$

$$= -\frac{\sqrt{3}}{2} \pm \frac{3}{2}\mathrm{i}$$

$$y = e^{-\frac{\sqrt{3}}{2}x}\left(A\cos\left(\frac{3x}{2}\right) + B\sin\left(\frac{3x}{2}\right)\right)$$

for any arbitrary real constants A and B

[3 marks]

Answer 5

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = 0$$

$$m^2 - 2m + 10 = 0 \quad \text{the auxiliary equation}$$

$$m = \frac{2 \pm \sqrt{4 - 4 \times 1 \times 10}}{2}$$

$$= 1 \pm 3\mathrm{i}$$

$$y = e^x\left(A\,cos\,(3x) + B\,sin\,(3x)\right)$$

for any arbitrary real constants A and B

As $y = 0$ when $x = 0$,

$$0 = e^0\left(A\,cos\,(0) + B\,sin\,(0)\right)$$

$$0 = A$$

$$\therefore \quad y = e^x\,(B\,sin\,(3x))$$

As $\dfrac{dy}{dx} = 3$ when $x = 0$,

$$\frac{dy}{dx} = 3B\,e^x\,cos\,(3x) + B\,e^x\,sin\,(3x)$$

$$3 = 3B\,e^0\,cos\,(0) + 2\,e^0\,sin\,(0)$$

$$3 = 3B$$

$$B = 1$$

The particular solution is thus,

$$y = e^x\,sin\,(3x)$$

[8 marks]

Lesson 5

Further A-Level Pure Mathematics, Core 2
Differential Equations II

5.1 Consolidation #1

Here is a summary of the previous four lessons. There is more to come but it's important to consolidate the techniques covered so far.

- Separating the variables can solve some first-order differential equations.

- A first-order differential equation of the form $\dfrac{dy}{dx} + P(x)\,y = Q(x)$ can be solved by multiplying every term by the integrating factor $I = e^{\int P(x)\,dx}$

- The nature of the roots α and β of the auxiliary equation determine the general solution to the second order differential equation $a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = 0$

  The general solution depends on the auxiliary equation's discriminant, D;

  ◇ Case 1, $\boldsymbol{D > 0}$

  Two distinct real roots: $y = A\,e^{\alpha x} + B\,e^{\beta x}$, for arbitrary constants, A, B.

  ◇ Case 2, $\boldsymbol{D = 0}$

  One repeated root: $y = (A + Bx)\,e^{\alpha x}$, for arbitrary constants, A, B.

  ◇ Case 3, $\boldsymbol{D < 0}$

  Two complex conjugate roots $\alpha = p + q\text{i}$, $\beta = p - q\text{i}$

  $y = e^{px}\,(A\cos qx + B\sin qx)$, for arbitrary constants, A, B.

5.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 38

Question 1

Given that $y = 1$ when $x = 0$, find the particular solution to the differential equation,

$$\frac{dy}{dx} = y\sinh x$$

[4 marks]

Question 2

(i) Let $M = \int e^{-3x} \sin x \; dx$

By using integration by parts twice show that,

$10\,M = -e^{-3x}(\cos x + 3\sin x) + c$, for some constant c

[4 marks]

(ii) $\dfrac{dy}{dx} - 3y = \sin x$

Given that $y = 0$ when $x = 0$, find y in terms of x.

[4 marks]

Question 3

Find the solution to the differential equation,

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = 0$$

given that, when $x = 0$, both $y = 0$ and $\frac{dy}{dx} = 3$

[8 marks]

Question 4

Find y in terms of k and x, given that

$$\frac{d^2y}{dx^2} + k^2 y = 0, \text{ where } k \text{ is a constant}$$

and $y = 1$ and $\frac{dy}{dx} = 1$ at $x = 0$

[8 marks]

Question 5

(i) Find the general solution to the differential equation,

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 0$$

[4 marks]

(ii) Given that $x = 1$ and $\frac{dx}{dt} = 1$ at $t = 0$, find the particular solution to the differential equation, giving your answer in the form $x = f(t)$

[2 marks]

(iii) Write your part (ii) answer in the form $R\,e^{kt}\cos(2t - \alpha)$ where k, R and α are constants that you have determined the exact value of.

[2 marks]

(iv) Sketch the curve with equation $x = f(t)$, $0 \leqslant t \leqslant \pi$, showing the coordinates, as multiples of π, of the points where the curve cuts the t-axis.

[2 marks]

5.3 Answers to 5.2 Exercise

Answer 1

$$\frac{dy}{dx} = y \sinh x$$

Separate the variables

$$\frac{1}{y}\frac{dy}{dx} = \sinh x$$

Integrate both sides with respect to x

$$\int \frac{1}{y}\, dy = \int \sinh x \; dx$$

$$\ln |y| = \cosh x + c$$

$$e^{\ln|y|} = e^{\cosh x + c}$$

$$|y| = e^{\cosh x + c}$$

But the exponential function is always positive and so,

$$y = e^{\cosh x + c}$$

Using the initial conditions that $y = 1, \; x = 0$

$$1 = e^{1+c} \quad \text{gives} \quad c = -1$$

$\therefore$ The particular solution is $y = e^{\cosh x - 1}$

[4 marks]

Answer 2

(i) $M = \int e^{-3x} \sin x \; dx$

$$= e^{-3x}(-\cos x) - \int (-3)\, e^{-3x}(-\cos x)\; dx$$

$$= -e^{-3x}\cos x - 3\int e^{-3x}\cos x \, dx + c_1$$

$$= -e^{-3x}\cos x - 3\left[e^{-3x}\sin x - \int (-3)\, e^{-3x}\sin x \, dx \right] + c_2$$

$$= -e^{-3x}\cos x - 3\,e^{-3x}\sin x - 9\int e^{-3x}\sin x \, dx + c_2$$

$$= -e^{-3x}\cos x - 3\,e^{-3x}\sin x - 9\,M + c$$

$$10\,M = -e^{-3x}\cos x - 3\,e^{-3x}\sin x$$

$$= -e^{-3x}(\cos x + 3\sin x) + c \qquad \square$$

[4 marks]

(ii) $\dfrac{dy}{dx} - 3y = \sin x$

$$P(x) = -3 \text{ and } Q(x) = \sin x$$

$$\int P(x)\ dx = \int (-3)\ dx = -3x + c$$

(constant of integration safely omitted)

$$I = e^{\int P(x)\,dx} = e^{-3x}$$

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = e^{-3x}\sin x \qquad \text{Multiplying through by } I$$

Integrate both sides with respect to x

$$\int e^{-3x}\frac{dy}{dx} - 3e^{-3x}y\ dx = \int e^{-3x}\sin x\ dx$$

$$e^{-3x}y = M \qquad \text{(About to use part (i))}$$

$$= \frac{1}{10}(-e^{-3x}(\cos x + 3\sin x) + c)$$

$$= -\frac{e^{-3x}}{10}(\cos x + 3\sin x) + A$$

$$y = Ae^{3x} - \frac{1}{10}(\cos x + 3\sin x)$$

As $y = 0$ when $x = 0$: $0 = A \times 1 - \dfrac{1}{10}(1 + 3 \times 0)$

$$A = \frac{1}{10}$$

$$\therefore y = \frac{e^{3x}}{10} - \frac{1}{10}(\cos x + 3\sin x)$$

[4 marks]

Answer 3

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = 0$$

$$m^2 - 2m + 10 = 0 \quad \text{is the auxiliary equation}$$

$$m = \frac{2 \pm \sqrt{4 - 4 \times 1 \times 10}}{2}$$

$$= 1 \pm 3\mathrm{i}$$

$$y = e^x\,(A\,cos\,(3x) + B\,sin\,(3x))$$

for any arbitrary real constants A and B

From $y = 0$ when $x = 0$ we get,

$$0 = 1\,(A \times 1 + B \times 0\,)$$

$$A = 0$$

So we now have that,

$$y = B\,e^x\,sin\,3x$$

$$\frac{dy}{dx} = 3\,B\,e^x\,cos\,3x + B\,e^x\,sin\,3x$$

$$= B\,e^x\,(\,3\,cos\,3x + sin\,3x\,)$$

From $\frac{dy}{dx} = 3$ when $x = 0$,

$$3 = B\,(3)$$

$$B = 1$$

$\therefore$ The particular solution is, $y = e^x\,sin\,3x$

[8 marks]

Answer 4

$$\frac{d^2y}{dx^2} + k^2 y = 0$$

$$m^2 + k^2 = 0 \quad \text{is the auxiliary equation}$$

$$m = \frac{0 \pm \sqrt{0 - 4 \times 1 \times k^2}}{2}$$

$$= \frac{0 \pm \sqrt{(-1)(2k)^2}}{2}$$

$$= 0 \pm k\mathrm{i}$$

$$y = e^{0x}(A\cos(kx) + B\sin(kx))$$

$$= A\cos(kx) + B\sin(kx)$$

for any arbitrary real constants A and B

From $y = 1$ when $x = 0$ we get,

$$1 = A\cos(0) + B\sin(0)$$

$$A = 1$$

So we now have that,

$$y = \cos(kx) + B\sin(kx)$$

$$\frac{dy}{dx} = -k\sin(kx) + Bk\cos(kx)$$

From $\frac{dy}{dx} = 1$ when $x = 0$,

$$1 = -k\sin(0) + Bk\cos(0)$$

$$1 = Bk$$

$$B = \frac{1}{k}$$

$\therefore$ The particular solution is, $y = \cos(kx) + \frac{1}{k}\sin(kx)$

[8 marks]

Answer 5

(i) $$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 0$$

$m^2 + 2m + 5 = 0$ is the auxiliary equation

$$m = \frac{-2 \pm \sqrt{4 - 4 \times 1 \times 5}}{2}$$

$$= \frac{-2 \pm \sqrt{(-1)(16)}}{2}$$

$$= -1 \pm 2i$$

$$x = e^{-t}(A\cos(2t) + B\sin(2t))$$

for any arbitrary real constants A and B

[4 marks]

(ii) From $x = 1$ when $t = 0$ we get,

$$1 = e^0(A\cos(0) + B\sin(0))$$

$$A = 1$$

So we now have that,

$$x = e^{-t}(\cos(2t) + B\sin(2t))$$

$$\frac{dx}{dt} = e^{-t}((-2)\sin(2t) + 2B\cos(2t)) - e^{-t}(\cos(2t) + B\sin(2t))$$

$$= e^{-t}(-2 - B)\sin(2t) + e^{-t}(2B - 1)\cos(2t)$$

From $\frac{dx}{dt} = 1$ when $t = 0$,

$$1 = (2B - 1)$$

$$B = 1$$

$\therefore$ The particular solution is, $x = e^{-t}(\cos(2t) + \sin(2t))$

[4 marks]

(iii) $\cos(2t - \alpha) = \cos 2t\cos\alpha + \sin 2t\sin\alpha$

We have $\cos(2t) \times 1 + \sin(2t) \times 1$

So $\cos\alpha = 1$ and $\sin\alpha = 1$

$\therefore \tan\alpha = \frac{\sin\alpha}{\cos\alpha} = 1$ giving $\alpha = \frac{\pi}{4}$

Also, $R = \sqrt{1^2 + 1^2} = \sqrt{2}$

$$x = e^{-t}(\cos(2t) + \sin(2t))$$

$$= \sqrt{2}\, e^{-t}\cos\left(2t - \frac{\pi}{4}\right), \qquad k = -1, R = \sqrt{2}, \alpha = \frac{\pi}{4}$$

[2 marks]

(iv) The t-axis crossing points will be given by $cos\left(2t - \frac{\pi}{4}\right) = 0$

$$2t - \frac{\pi}{4} = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

$$2t = \frac{3\pi}{4}, \frac{7\pi}{4}, \ldots$$

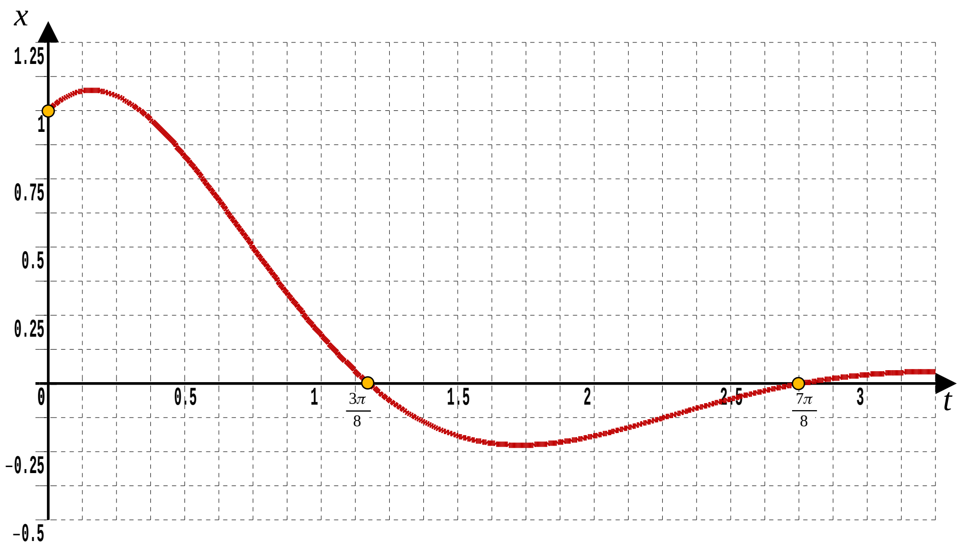
$$t = \frac{3\pi}{8}, \frac{7\pi}{8}\ldots.$$

The x-axis crossing point is given when $t = 0$,

$$x = \sqrt{2}\, e^0 cos\left(0 - \frac{\pi}{4}\right)$$

$$= 1$$

From part (iii) you know the curve is essentially a cosine wave that has been translated $\begin{pmatrix} \frac{\pi}{4} \\ 0 \end{pmatrix}$ but its dampened by a decaying exponential envelope. Here's the accurate graph from by graph-plotter.



6.1 Complementary Function + Particular Integral

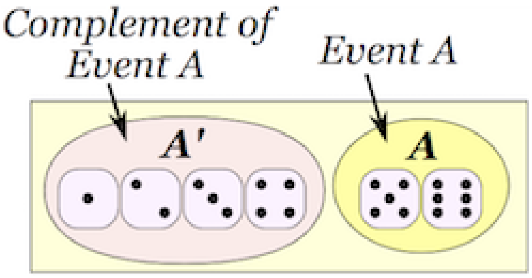
Via consideration of the discriminant of the auxiliary equation, we have developed techniques to solve second-order, linear, homogeneous, differential equations with constant coefficients. That is, equations of the form,

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

It is the zero on the right hand side that earned it the descriptor “homogeneous”. With not much extra work, we can extend our repertoire to handle the situation where the zero is replaced with a function of x. In other words, “nonhomogeneous” differential equations of the form,

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

To solve this type of nonhomogeneous equation, first find the general solution of the corresponding homogeneous differential equation. This general solution is called the complementary function, CF.



The hunt is then on to find a particular integral, PI, which is a function that satisfied the nonhomogeneous differential equation. Here we take the advice of mathematicians that have gone before us. Their collective wisdom is contained in the following table. Depending on the nature of $f(x)$ they suggest what form the particular integral will take.

Form of $f(x)$	Form of PI
u	U
$ux + v$	$Ux + V$
$u x^r + \dots + vx + w$	$U x^r + \dots + Vx + W$
$u \cos kx$	$U \cos kx + V \sin kx$
$v \sin kx$	$U \cos kx + V \sin kx$
$u \cos kx + v \sin kx$	$U \cos kx + V \sin kx$
$u e^{kx}$	$U e^{kx}$
$u e^{-kx}$	$U e^{-kx}$

The general solution to the nonhomogeneous equation is then CF + PI

6.2 Example

Find the general solution to the differential equation,

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 12y = 36x$$

[6 marks]

6.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1

Find the general solution to the differential equation,

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 6$$

[6 marks]

Question 2

Find the general solution to the differential equation,

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 5e^{3x}$$

[6 marks]

Question 3

Find the general solution to the differential equation,

$$\frac{d^2y}{dx^2} - y = 2x^2 - x - 3$$

[6 marks]

Question 4

Find the general solution to the differential equation,

$$\frac{d^2y}{dx^2} + 4y = \sin x$$

[6 marks]

Question 5

Find the general solution to the differential equation,

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 25\cos 2x$$

[7 marks]

Question 6

Further A-Level Examination Question from June 2012, Paper FP2, Q4 (Edexcel)

Find the general solution to the differential equation,

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 2\cos t - \sin t$$

[9 marks]

6.4 Answers

6.4.1 Solution to 6.2 Example

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 12y = 36x$$

Obtain the complementary function by solving the homogeneous equation,

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 12y = 0$$

$$m^2 - 8m + 12 = 0 \quad \text{is the auxiliary equation}$$

$$(m - 6)(m - 2) = 0$$

Either $m = 6$ or $m = 2$

We have two distinct real roots

$\therefore$ The complementary function is $y = Ae^{2x} + Be^{6x}$

For a particular integral try $y = Ux + V$ (note the "V" is needed)

in which case $\dfrac{dy}{dx} = U$ and $\dfrac{d^2y}{dx^2} = 0$

Substitute these into the question's differential equation...

$$0 - 8U + 12(Ux + V) = 36x$$

$$-8U + 12Ux + 12V = 36x$$

Coefficients of x : $12U = 36$ gives $U = 3$

Terms independent of x : $-8U + 12V = 0$ gives $V = 2$

$\therefore$ The particular integral is $y = 3x + 2$

The general solution is $y = Ae^{2x} + Be^{6x} + 3x + 2$

[6 marks]

6.4.2 Solution to 6.3 Exercise

Answer 1

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 6$$

Obtain the complementary function by solving the homogeneous equation,

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0$$

$$m^2 - 2m - 3 = 0 \quad \text{is the auxiliary equation}$$

$$(m + 1)(m - 3) = 0$$

$$\text{Either } m = -1 \text{ or } m = 3$$

We have two distinct real roots

$\therefore$ The complementary function is $y = A e^{-x} + B e^{3x}$

For a particular integral try $y = U$

$$\text{in which case } \frac{dy}{dx} = 0 \text{ and } \frac{d^2y}{dx^2} = 0$$

Substitute these into the question's differential equation...

$$0 - 0 - 3U = 6$$

$$U = -2$$

$\therefore$ The particular integral is $y = -2$

The general solution is $y = A e^{-x} + B e^{3x} - 2$

[6 marks]

Answer 2

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 5e^{3x}$$

Obtain the complementary function by solving the homogeneous equation,

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

$m^2 - 3m + 2 = 0$ is the auxiliary equation

$$(m - 1)(m - 2) = 0$$

Either $m = 1$ or $m = 2$

We have two distinct real roots

$\therefore$ The complementary function is $y = Ae^{x} + Be^{2x}$

For a particular integral try $y = Ue^{3x}$

in which case $\frac{dy}{dx} = 3Ue^{3x}$ and $\frac{d^2y}{dx^2} = 9Ue^{3x}$

Substitute these into the question's differential equation...

$$9Ue^{3x} - 3 \times 3Ue^{3x} + 2Ue^{3x} = 5e^{3x}$$

$$9U - 9U + 2U = 5$$

$$U = 2.5$$

$\therefore$ The particular integral is $y = 2.5e^{3x}$

The general solution is $y = Ae^{x} + Be^{2x} + 2.5e^{3x}$

[6 marks]

Answer 3

$$\frac{d^2y}{dx^2} - y = 2x^2 - x - 3$$

Obtain the complementary function by solving the homogeneous equation,

$$\frac{d^2y}{dx^2} - y = 0$$

$$m^2 - 1 = 0 \text{ is the auxiliary equation}$$

$$(m + 1)(m - 1) = 0$$

Either $m = -1$ or $m = 1$

We have two distinct real roots

$\therefore$ The complementary function is $y = Ae^{-x} + Be^{x}$

For a particular integral try $y = Ux^2 + Vx + W$

in which case $\dfrac{dy}{dx} = 2Ux + V$ and $\dfrac{d^2y}{dx^2} = 2U$

Substitute these into the question's differential equation...

$$2U - Ux^2 - Vx - W = 2x^2 - x - 3$$

Matching coefficients of x^2 : $U = -2$

Matching coefficients of x : $V = 1$

Terms independent of x : $2U - W = -3$ gives $W = -1$

$\therefore$ The particular integral is $y = -2x^2 + x - 1$

The general solution is $y = Ae^{-x} + Be^{x} - 2x^2 + x - 1$

[6 marks]

Answer 4

$$\frac{d^2y}{dx^2} + 4y = \sin x$$

Obtain the complementary function by solving the homogeneous equation,

$$\frac{d^2y}{dx^2} + 4y = 0$$

$m^2 + 4 = 0$ is the auxiliary equation

$$m^2 - (2\mathrm{i})^2 = 0$$

$(m + 2\mathrm{i})(m - 2\mathrm{i}) = 0$ difference of two squares

Either $m = -2\mathrm{i}$ or $m = 2\mathrm{i}$

We have two complex roots

$\therefore$ The complementary function is $y = A\cos 2x + B\sin 2x$

For a particular integral try $y = U\cos x + V\sin x$

in which case $\dfrac{dy}{dx} = -U\sin x + V\cos x$

and $\dfrac{d^2y}{dx^2} = -U\cos x - V\sin x$

Substitute these into the question's differential equation...

$$-U\cos x - V\sin x + 4U\cos x + 4V\sin x = \sin x$$

Matching coefficients of $\cos x$: $-U + 4U = 0$ giving $U = 0$

Matching coefficients of $\sin x$: $-V + 4V = 1$ giving $V = \dfrac{1}{3}$

$\therefore$ The particular integral is $y = \dfrac{1}{3}\sin x$

The general solution is $y = A\cos 2x + B\sin 2x + \dfrac{1}{3}\sin x$

[6 marks]

Answer 5

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 25\cos 2x$$

Obtain the complementary function by solving the homogeneous equation,

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

$m^2 + 2m + 1 = 0$ is the auxiliary equation

$$(m + 1)^2 = 0$$

$$m = -1$$

We have a repeated root

$\therefore$ The complementary function is $y = (A + Bx)e^{-x}$

For a particular integral try $y = U\cos 2x + V\sin 2x$

in which case $\frac{dy}{dx} = -2U\sin 2x + 2V\cos 2x$

and $\frac{d^2y}{dx^2} = -4U\cos 2x - 4V\sin 2x$

Substitute these into the question's differential equation...

$$-4U\cos 2x - 4V\sin 2x - 4U\sin 2x + 4V\cos 2x + U\cos 2x + V\sin 2x = 25\cos 2x$$

Matching coefficients of $\cos 2x$: $-4U + 4V + U = 25$

$$-3U + 4V = 25$$

Matching coefficients of $\sin 2x$: $-4V - 4U + V = 0$

$$-4U - 3V = 0$$

$$-9U + 12V = 75$$

$$ADD \quad -16U - 12V = 0$$

$$-25U = 75$$

$$U = -3 \text{ and } V = 4$$

$\therefore$ The particular integral is $y = -3\cos 2x + 4\sin 2x$

The general solution is $y = (A + Bx)e^{-x} - 3\cos 2x + 4\sin 2x$

[7 marks]

Answer 6

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 2\cos t - \sin t$$

Obtain the complementary function by solving the homogeneous equation,

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0$$

$m^2 + 5m + 6 = 0$ is the auxiliary equation

$$(m+3)(m+2) = 0$$

Either $m = -3$ or $m = -2$

We have two distinct real roots

$\therefore$ The complementary function is $x = Ae^{-3t} + Be^{-2t}$

For a particular integral try $x = U\cos t + V\sin t$

in which case $\dfrac{dx}{dt} = -U\sin t + V\cos t$

and $\dfrac{d^2x}{dt^2} = -U\cos t - V\sin t$

Substitute these into the question's differential equation...

$$-U\cos t - V\sin t - 5U\sin t + 5V\cos t + 6U\cos t + 6V\sin t = 2\cos t - \sin t$$

Matching coefficients of $\cos t$: $\quad -U + 5V + 6U = 2$

$$5U + 5V = 2$$

Matching coefficients of $\sin t$: $\quad -V - 5U + 6V = -1$

$$-5U + 5V = -1$$

$$\begin{array}{rrcr} & 5U + 5V & = & 2 \\ ADD & -5U + 5V & = & -1 \\ \hline & 10V & = & 1 \end{array}$$

$$V = \frac{1}{10} \text{ and } U = \frac{3}{10}$$

$\therefore$ The particular integral is $y = \dfrac{3}{10}\cos t + \dfrac{1}{10}\sin t$

The general solution is $y = Ae^{-3t} + Be^{-2t} + \dfrac{3}{10}\cos t + \dfrac{1}{10}\sin t$

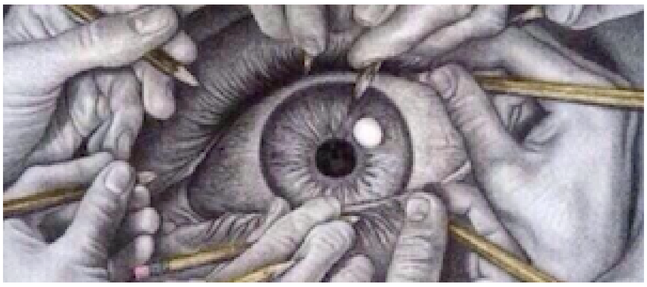
[9 marks]

7.1 Dark Art Particular Integrals

Previously, the following table of suggested particular integrals was given. It has working well with the second order nonhomogeneous differential equations so far considered.

Form of $f(x)$	Form of PI
u	U
$ux + v$	$Ux + V$
$u x^r + \ldots + vx + w$	$U x^r + \ldots + Vx + W$
$u \cos kx$	$U \cos kx + V \sin kx$
$v \sin kx$	$U \cos kx + V \sin kx$
$u \cos kx + v \sin kx$	$U \cos kx + V \sin kx$
$u e^{kx}$	$U e^{kx}$
$u e^{-kx}$	$U e^{-kx}$

However, finding a particular integral can sometimes be something of a dark art and what a skilled mathematician chooses may not be obvious to beginners.



There is one additional piece of "wisdom from the elders" that we will embrace.

Clash of Function : An "Advice" Algorithm

When solving an equation of the form,

$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = f(x), \text{ where } a, b \text{ and } c \text{ are constants,}$$

and having obtained the complementary function,

if a piece, $g(x)$, of a proposed particular integral, $p(x)$, has already occurred in the complementary function, modify the particular integral by multiplying it by x. That is, replace $p(x)$ with $x\,p(x)$.

Repeat this process, if necessary, until there is no piece in common in any part of the complementary function or particular integral.

Note: this advice algorithm was already at work within the complementary function when a repeated root occurred.

7.2 Example

Find the general solution to the differential equation,

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} = 12$$

[6 marks]

What do you call the sudden urge to solve a differential equation ?

Calculust !

7.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 40

Question 1

Find the general solution to the differential equation,

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = e^{-3x}$$

[8 marks]

Question 2

y satisfies the differential equation,

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} = 2x^2 - x + 1$$

(i) Find the complementary function for this differential equation.

[3 marks]

(ii) Explain, with reference to your part (i) answer, why the "obvious" particular integral of $y = Ux^2 + Vx + W$ is not suitable.

[2 marks]

(iii) By using a suitable particular integral, find the general solution.

[8 marks]

Question 3

Find the general solution to the differential equation,

$$\frac{d^2y}{dx^2} + 12\,\frac{dy}{dx} + 36y = 6\,e^{-6x}$$

[9 marks]

Question 4

Find the general solution to the differential equation,

$$\frac{d^2y}{dx^2} + 16y = \cos 4x$$

[10 marks]

7.4 Answers

7.4.1 Solution to 7.2 Example

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} = 12$$

Obtain the complementary function by solving the homogeneous equation,

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} = 0$$

$$m^2 - 4m = 0 \quad \text{is the auxiliary equation}$$

$$m(m - 4) = 0$$

$$\text{Either } m = 0 \text{ or } m = 4$$

We have two distinct real roots

$\therefore$ The complementary function is $y = Ae^{0x} + Be^{4x}$

$$y = A + Be^{4x}$$

For a particular integral we'd normally try $y = U$ but this function of a constant is already in the complementary function.

Instead, try $y = Ux$ in which case $\frac{dy}{dx} = U$ and $\frac{d^2y}{dx^2} = 0$

Substitute these into the question's differential equation...

$$0 - 4U = 12$$

$$U = -3$$

$\therefore$ The particular integral is $y = -3x$

The general solution is $y = A + Be^{4x} - 3x$

[6 marks]

7.4.2 Solutions to 7.3 Exercise

Answer 1

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = e^{-3x}$$

Obtain the complementary function by solving the homogeneous equation,

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

$$m^2 + m - 6 = 0 \quad \text{is the auxiliary equation}$$

$$(m + 3)(m - 2) = 0$$

$$\text{Either } m = -3 \text{ or } m = 2$$

We have two distinct real roots

$\therefore$ The complementary function is $y = A e^{-3x} + B e^{2x}$

For a particular integral we'd normally try $y = Ue^{-3x}$ but the function $\left(\text{a multiple of } e^{-3x}\right)$ is already in the complementary function.

Instead, try $y = Ux\, e^{-3x}$ in which case,

$$\frac{dy}{dx} = Ux(-3)e^{-3x} + Ue^{-3x}$$

$$\text{and } \frac{d^2y}{dx^2} = (-3Ux)(-3)e^{-3x} + (-3U)e^{-3x} + (-3)Ue^{-3x}$$

$$= 9Ux\, e^{-3x} - 6Ue^{-3x}$$

Substitute these into the question's differential equation...

$$9Ux\, e^{-3x} - 6Ue^{-3x} - 3Ux\, e^{-3x} + Ue^{-3x} - 6Ux\, e^{-3x} = e^{-3x}$$

$$-5U = 1$$

$$U = -\frac{1}{5}$$

$\therefore$ The particular integral is $y = -\frac{1}{5} x\, e^{3x}$

The general solution is $y = A e^{-3x} + B e^{2x} - \frac{1}{5} x\, e^{3x}$

[8 marks]

Answer 2

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} = 2x^2 - x + 1$$

(i) Obtain the complementary function by solving the homogeneous equation,

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} = 0$$

$m^2 - 6m = 0$ is the auxiliary equation

$$m(m - 6) = 0$$

Either $m = 0$ or $m = 6$

We have two distinct real roots

$\therefore$ The complementary function is $y = Ae^{0x} + Be^{6x}$

$$= A + Be^{6x}$$

[3 marks]

(ii) For a particular integral we'd normally try $y = Ux^2 + Vx + W$ but there is already a constant in the part (i) answer's complementary function.

[2 marks]

(iii) Instead, try $y = Ux^3 + Vx^2 + Wx$ in which case,

$$\frac{dy}{dx} = 3Ux^2 + 2Vx + W$$

and $\frac{d^2y}{dx^2} = 6Ux + 2V$

Substitute these into the question's differential equation...

$$6Ux + 2V - 18Ux^2 - 12Vx - 6W = 2x^2 - x + 1$$

$$-18Ux^2 + (6U - 12V)x + 2V - 6W = 2x^2 - x + 1$$

Matching coefficients of x^2 : $-18U = 2$ giving $U = -\frac{1}{9}$

Matching coefficients of x : $6U - 12V = -1$ giving $V = \frac{1}{36}$

Terms independent of x : $2V - 6W = 1$ gives $W = -\frac{17}{108}$

$\therefore$ The particular integral is $y = -\frac{1}{9}x^3 + \frac{1}{36}x^2 - \frac{17}{108}x$

The general solution is $y = A + Be^{6x} - \frac{1}{9}x^3 + \frac{1}{36}x^2 - \frac{17}{108}x$

[8 marks]

Answer 3

$$\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 36y = 6e^{-6x}$$

Obtain the complementary function by solving the homogeneous equation,

$$\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 36y = 0$$

$$m^2 + 12m + 36 = 0 \quad \text{is the auxiliary equation}$$

$$(m + 6)^2 = 0$$

$$m = -6$$

We have repeated roots

$\therefore$ The complementary function is $y = (A + Bx)\,e^{-6x}$

For a particular integral we'd normally try $y = Ue^{-6x}$ but that function $\left(\text{a multiple of } e^{-6x}\right)$ is already in the complementary function.

Instead we start to consider $y = Ux\,e^{-6x}$ but this too causes a clash.

Thus we end up trying $y = Ux^2 e^{-6x}$

$$\frac{dy}{dx} = Ux^2(-6)e^{-6x} + 2Ux\,e^{-6x}$$

$$\text{and} \quad \frac{d^2y}{dx^2} = \left(-6Ux^2\right)(-6)e^{-6x} + (-12Ux)e^{-6x}$$

$$+ (2Ux)(-6)Ue^{-6x} + 2Ue^{-6x}$$

$$= 36Ux^2 e^{-6x} - 24Ux\,e^{-6x} + 2Ue^{-6x}$$

Substitute these into the question's differential equation...

$$36Ux^2 e^{-6x} - 24Ux\,e^{-6x} + 2Ue^{-6x}$$

$$- 72Ux^2e^{-6x} + 24Ux\,e^{-6x} + 36\,Ux^2 e^{-6x} = 6e^{-6x}$$

$$2Ue^{-6x} = 6e^{-6x}$$

$$U = 3$$

$\therefore$ The particular integral is $y = 3x^2 e^{-6x}$

The general solution is $y = \left(A + Bx + 3x^2\right)e^{-6x}$

[9 marks]

Answer 4

$$\frac{d^2y}{dx^2} + 16y = \cos 4x$$

Obtain the complementary function by solving the homogeneous equation,

$$\frac{d^2y}{dx^2} + 16y = 0$$

$$m^2 + 16 = 0 \quad \text{is the auxiliary equation}$$

$$m^2 - (4\mathrm{i})^2 = 0$$

$$(m + 4\mathrm{i})(m - 4\mathrm{i}) = 0 \quad \text{difference of two squares}$$

Either $m = -4\mathrm{i}$ or $m = 4\mathrm{i}$ (We have two complex roots)

$\therefore$ The complementary function is $y = A\cos 4x + B\sin 4x$

For a particular integral we'd normally try $y = U\cos 4x + V\sin 4x$ but a piece of this $\left(\text{a multiple of } \cos 4x\right)$ is already in the complementary function.

Instead try, $\quad y = Ux\cos 4x + Vx\sin 4x$

in which case $\dfrac{dy}{dx} = Ux(-4\sin 4x) + U\cos 4x + Vx\,4\cos 4x + V\sin 4x$

$$= (-4Ux + V)\sin 4x + (U + 4Vx)\cos 4x$$

and $\dfrac{d^2y}{dx^2} = (-4Ux + V)(4\cos 4x) + (-4U)\sin 4x$

$$+ (U + 4Vx)(-4\sin 4x) + (4V)\cos 4x$$

$$= (-16Ux + 8V)\cos 4x + (-8U - 16Vx)\sin 4x$$

Substitute these into the question's differential equation...

$$(-16Ux + 8V)\cos 4x + (-8U - 16Vx)\sin 4x + 16Ux\cos 4x + 16Vx\sin 4x = \cos 4x$$

$$(-16Ux + 8V + 16Ux)\cos 4x + (-8U - 16Vx + 16Vx)\sin 4x = \cos 4x$$

$$8V\cos 4x - 8U\sin 4x = \cos 4x$$

Matching coefficients of $\cos x$: $8V = 1$ giving $V = \dfrac{1}{8}$

Matching coefficients of $\sin x$: $8U = 0$ giving $U = 0$

$\therefore$ The particular integral is $y = \dfrac{1}{8}x\sin 4x$

The general solution is $y = A\cos 4x + B\sin 4x + \dfrac{1}{8}x\sin 4x$

[10 marks]

8.2 Consolidation #2

Abbreviated summary;

- Separating the variables can solve some first-order differential equations.

- A first-order differential equation of the form $\frac{dy}{dx} + P(x)\,y = Q(x)$ can be solved by multiplying every term by the integrating factor $I = e^{\int P(x)\,dx}$

- The nature of the roots α and β of the auxiliary equation determine the general solution to the second order differential equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$
  The general solution depends on the auxiliary equation's discriminant, D;
  ◇ Case 1, $\boldsymbol{D > 0}$: $y = Ae^{\alpha x} + Be^{\beta x}$, for arbitrary A, B.
  ◇ Case 2, $\boldsymbol{D = 0}$: $y = (A + Bx)\,e^{\alpha x}$, for arbitrary A, B.
  ◇ Case 3, $\boldsymbol{D < 0}$: $y = e^{px}\,(A\cos qx + B\sin qx)$, for arbitrary A, B.

- Particular Integral suggestions,

Form of $f(x)$	Form of PI
$u\,x^r + \ldots + vx + w$	$U\,x^r + \ldots + Vx + W$
$u\cos kx + v\sin kx$	$U\cos kx + V\sin kx$
$u\,e^{kx}$	$U\,e^{kx}$

- But watch out for...

Clash of Function : An “Advice” Algorithm

When solving an equation of the form,

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x), \text{ where } a, b \text{ and } c \text{ are constants,}$$

and having obtained the complementary function,
if a piece, $g(x)$, of a proposed particular integral, $p(x)$, has already occurred in the complementary function, modify the particular integral by multiplying it by x. That is, replace $p(x)$ with $x\,p(x)$.
Repeat this process, if necessary, until there is no piece in common in any part of the complementary function or particular integral.

8.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 50

Question 1

Find the general solution to the differential equation,

$$y'' + 2\sqrt{2}\, y' + 2y = 10$$

[5 marks]

Question 2

Find the general solution to the differential equation,

$$y'' - y' - 12y = 144x$$

[7 marks]

Question 3

Find the general solution to the differential equation,

$$y'' - 7y' + 10y = x\,e^{x}$$

Assume the particular integral is of the form $(Ux + V)\,e^{x}$

[10 marks]

Question 4

(i) Find the value of U for which $y = Ue^{2x}$ is a particular integral of the differential equation $y'' - 4y' + 13y = e^{2x}$

[4 marks]

(ii) Using your answer to part (i), find the general solution to the differential equation.

[5 marks]

Question 5

The differential equation $y'' - 4y' + 4y = 4e^{2x}$ is to be solved.

(i) Find the complementary function.

[3 marks]

(ii) Explain why neither Ue^{2x} nor $Ux\,e^{2x}$ can be a particular integral.

[2 marks]

A particular integral has the form $Ux^2\,e^{2x}$

(iii) Determine the value of the constant U and find the general solution.

[6 marks]

Question 6

Cameron, who is skilled in the dark arts, is about to advise you on the particular integral to use when solving the differential equation,

$$y'' + y = 3\,sin\,2x$$

Although your natural inclination is to assume a particular integral of the form $y = U sin\,2x + V cos\,2x$, Cameron assures you that $U sin\,2x$ is sufficient.

(i) Given that Cameron speaks wise words, determine the value of *U*.

[4 marks]

(ii) Using your answer to part (i), find the general solution to the differential equation.

[4 marks]

8.3.1 Solutions to 8.2 Exercise

Answer 1

$$y'' + 2\sqrt{2}\,y' + 2y = 10$$

Obtain the complementary function by solving the homogeneous equation,

$$y'' + 2\sqrt{2}\,y' + 2y = 0$$

$$m^2 + 2\sqrt{2}\,m + 2 = 0 \quad \text{is the auxiliary equation}$$

$$\left(m + \sqrt{2}\right)^2 = 0$$

$$m = -\sqrt{2}$$

We have a repeated root

$\therefore$ The complementary function is $y = (A + Bx)\,e^{-\sqrt{2}\,x}$

For the particular integral, $y = U$, giving $y' = y'' = 0$

Substitute these into the question's differential equation...

$$0 + 2 \times 0 + 2U = 10$$

$$U = 5$$

$\therefore$ The particular integral is $y = 5$

The general solution is $y = (A + Bx)\,e^{-\sqrt{2}\,x} + 5$

[5 marks]

Answer 2

$$y'' - y' - 12y = 144x$$

Obtain the complementary function by solving the homogeneous equation,

$$y'' - y' - 12y = 0$$

$$m^2 - m - 12 = 0 \quad \text{is the auxiliary equation}$$

$$(m + 3)(m - 4) = 0$$

$$\text{Either } m = -3 \text{ or } m = 4$$

We have two distinct real roots

$\therefore$ The complementary function is $y = Ae^{-3x} + Be^{4x}$

For a particular integral try $y = Ux + V$ (note the "V" is needed)

in which case $y' = U$ and $y'' = 0$

Substitute these into the question's differential equation...

$$0 - U - 12(Ux + V) = 144x$$

$$-U - 12Ux - 12V = 144x$$

Coefficients of x : $-12U = 144$ gives $U = -12$

Terms independent of x : $-U - 12V = 0$ gives $V = 1$

$\therefore$ The particular integral is $y = -12x + 1$

The general solution is $y = Ae^{-3x} + Be^{4x} - 12x + 1$

[6 marks]

Answer 3

$$y'' - 7y' + 10y = x\,e^x$$

Obtain the complementary function by solving the homogeneous equation,

$$y'' - 7y' + 10y = 0$$

$$m^2 - 7m + 10 = 0 \text{ is the auxiliary equation}$$

$$(m - 2)(m - 5) = 0$$

$$\text{Either } m = 2 \text{ or } m = 5$$

We have two distinct real roots

$\therefore$ The complementary function is $y = A\,e^{2x} + B\,e^{5x}$

For the particular integral, $y = (Ux + V)\,e^x$

$$y' = (Ux + V)\,e^x + Ue^x$$

$$= (Ux + U + V)e^x$$

$$y'' = (Ux + U + V)e^x + Ue^x$$

$$= (Ux + 2U + V)e^x$$

Substitute these into the question's differential equation...

$$(Ux + 2U + V)e^x - 7(Ux + U + V)e^x + 10(Ux + V)\,e^x = x\,e^x$$

$$4Ux - 5U + 4V = x$$

Matching coefficients of x : $4U = 1$ giving $U = \dfrac{1}{4}$

Terms independent of x : $-\dfrac{5}{4} + 4V = 0$ giving $V = \dfrac{5}{16}$

$\therefore$ The particular integral is $y = \left(\dfrac{x}{4} + \dfrac{5}{16}\right)e^x$

The general solution is $y = A\,e^{2x} + B\,e^{5x} + \left(\dfrac{x}{4} + \dfrac{5}{16}\right)e^x$

[10 marks]

Answer 4

(i) $y = Ue^{2x} \Rightarrow y' = 2Ue^{2x} \Rightarrow y'' = 4Ue^{2x}$

Substituting these into $y'' - 4y' + 13y = e^{2x}$ gives,

$$4Ue^{2x} - 4 \times 2Ue^{2x} + 13 \times Ue^{2x} = e^{2x}$$

$$4U - 8U + 13U = 1$$

$$9U = 1$$

$$U = \frac{1}{9}$$

[4 marks]

(ii) Obtain the complementary function by solving the homogeneous equation,

$$y'' - 4y' + 13y = 0$$

$$m^2 - 4m + 13 = 0 \quad \text{is the auxiliary equation}$$

$$m = \frac{4 \pm \sqrt{16 - 4 \times 1 \times 13}}{2}$$

$$= \frac{4 \pm \sqrt{(-1)(36)}}{2}$$

$$= \frac{4 \pm 6\mathrm{i}}{2}$$

$$= 2 \pm 3\mathrm{i}$$

We have two complex roots

$\therefore$ The complementary function is $y = e^{2x}\,(A\cos 3x + B\sin 3x)$

The general solution is $y = e^{2x}\,(A\cos 3x + B\sin 3x) + \frac{1}{9}e^{2x}$

[4 marks]

Answer 5

(i) $y'' - 4y' + 4y = 4e^{2x}$

Obtain the complementary function by solving the homogeneous equation,

$$y'' - 4y' + 4y = 0$$

$$m^2 - 4m + 4 = 0 \quad \text{is the auxiliary equation}$$

$$(m - 2)^2 = 0$$

$$m = 2$$

We have a repeated root

$\therefore$ The complementary function is $y = (A + Bx)\, e^{2x}$

[3 marks]

(ii) Both Ue^{2x} and $Ux\, e^{2x}$ are part of the complementary function.

[2 marks]

(iii) For the particular integral $y = Ux^2 e^{2x}$

$$y' = Ux^2 (2) e^{2x} + 2Ux\, e^{2x}$$

$$= \left(2Ux^2 + 2Ux \right) e^{2x}$$

$$y'' = \left(2Ux^2 + 2Ux \right) 2\, e^{2x} + \left(4Ux + 2U \right) e^{2x}$$

$$= \left(4Ux^2 + 8Ux + 2U \right) e^{2x}$$

Substitute these into the question's differential equation...

$$\left(4Ux^2 + 8Ux + 2U \right) e^{2x} - 4\left(2Ux^2 + 2Ux \right) e^{2x} + 4Ux^2 e^{2x} = 4\, e^{2x}$$

$$2U = 4$$

$$U = 2$$

$\therefore$ The particular integral is $y = 2x^2 e^{2x}$

The general solution is $y = \left(A + Bx + 2x^2 \right) e^{2x}$

[6 marks]

Answer 6

(i) $y = U \sin 2x \Rightarrow y' = 2U \cos 2x \Rightarrow y'' = -4U \sin 2x$

Substituting these into $y'' + y = 3 \sin 2x$ gives,

$$-4U \sin 2x + U \sin 2x = 3 \sin 2x$$

$$-4U + U = 3$$

$$-3U = 3$$

$$U = -1$$

[4 marks]

(ii) Obtain the complementary function by solving the homogeneous equation,

$$y'' + y = 0$$

$m^2 + 1 = 0$ is the auxiliary equation

$$m^2 = -1$$

$$= \pm \mathrm{i}$$

We have two complex roots

$\therefore$ The complementary function is $y = e^{0x} (A \cos x + B \sin x)$

The general solution is $y = (A \cos x + B \sin x) - \sin 2x$

[4 marks]

Lesson 9

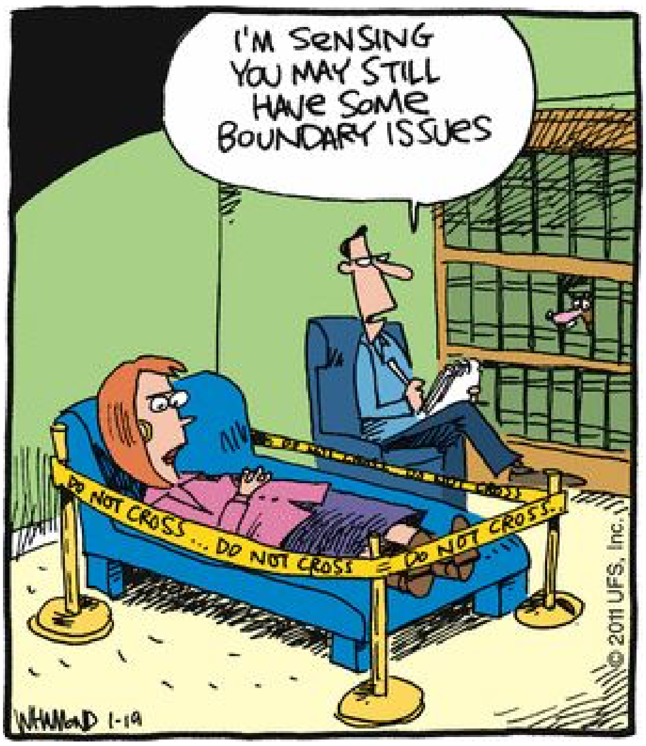
Further A-Level Pure Mathematics, Core 2
Differential Equations II

9.1 Boundary Conditions

The general solution to a second order differential equation typically involves some arbitrary constants that we have tended to label *A* and *B*. A question may provide some extra information, some boundary conditions, that allow the particular solution to be found, that is, to work out numerical values for *A* and *B*. Usually, two pieces of extra information will be given. Each piece may yield the value one of the arbitrary constants or you may need to set up and solve two simultaneous equations to get the values.

9.2 Example

Given that $y'(0) = 1$ and $y(0) = 0$, find the particular solution to the differential equation $y'' + 5y' + 4y = 4$



[8 marks]

9.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 50

Question 1

Given that $y'(0) = 18$ and $y(0) = 36$, find the particular solution to the differential equation $y'' + 7y' - 8y = 72$

[8 marks]

Question 2

Given that $y'(0) = 0$ and $y(0) = 4$, find the particular solution to the differential equation $y'' + 12y' + 37y = 10e^{-4x}$

[9 marks]

Question 3

Find y in terms of k and x, given that $\frac{d^2y}{dx^2} + k^2 y = 0$ where k is a constant, and $y = 1$ and $\frac{dy}{dx} = 1$ at $x = 0$

[8 marks]

Question 4

Further A-Level Examination Question from June 2014, FP2, Q5 (Edexcel)

(a) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 27e^{-x}$$

[6 marks]

(b) Find the particular solution that satisfies $y = 0$ and $\frac{dy}{dx} = 0$ when $x = 0$

[6 marks]

Question 5

Further A-Level Examination Question from June 2017, FP2, Q5 (Edexcel)

(a) Find the general solution of the differential equation,

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 26 \sin 3x$$

[8 marks]

(b) Find the particular solution of this differential equation for which $y = 0$ and $\frac{dy}{dx} = 0$ when $x = 0$

[5 marks]

9.4 Answers

9.4.1 Solutions to 9.2 Example

$$y'' + 5y' + 4y = 4$$

Obtain the complementary function by solving the homogeneous equation,

$$y'' + 5y' + 4y = 0$$

$$m^2 + 5m + 4 = 0 \quad \text{is the auxiliary equation}$$

$$(m + 4)(m + 1) = 0$$

$$\text{Either } m = -4 \text{ or } m = -1$$

Two distinct roots

$\therefore$ The complementary function is $y = Ae^{-4x} + Be^{-x}$

For the particular integral, $y = U$ so $y' = y'' = 0$

Substitute these into the question's differential equation...

$$0 + 5 \times 0 + 4U = 4$$

$$U = 1$$

The particular integral is $y = 2$

The general solution is $y = Ae^{-4x} + Be^{-x} + 1$

from which $y' = -4Ae^{-4x} - Be^{-x}$

From the boundary conditions $y'(0) = 1, \ y(0) = 0$

$$A + B + 1 = 0 \text{ and } -4A - B = 1$$

Solving these simultaneously...

$$\begin{array}{lrcl} & -4A - B &=& 1 \\ \text{ADD} & A + B &=& -1 \\ \hline & -3A &=& 0 \end{array}$$

$$\text{So } A = 0 \text{ and } B = -1$$

The particular solution is,

$$y = 1 - e^{-x}$$

[8 marks]

9.4.2 Solutions to 9.3 Exercise

Answer 1

$$y'' + 7y' - 8y = 72$$

Obtain the complementary function by solving the homogeneous equation,

$$y'' + 7y' - 8y = 0$$

$$m^2 + 7m - 8 = 0 \text{ is the auxiliary equation}$$

$$(m + 8)(m - 1) = 0$$

$$\text{Either } m = -8 \text{ or } m = 1$$

Two distinct roots

$\therefore$ The complementary function is $y = A e^{-8x} + B e^{x}$

For the particular integral, $y = U$ so $y' = y'' = 0$

Substitute these into the question's differential equation...

$$0 + 7 \times 0 - 8U = 72$$

$$U = -9$$

The particular Integral is $y = -9$

The general solution is $y = A e^{-8x} + B e^{x} - 9$

from which $y' = -8A e^{-8x} + B e^{x}$

From the boundary conditions $y'(0) = 18,\ y(0) = 36$

$$-8A + B = 18 \text{ and } A + B - 9 = 36$$

Solving these simultaneously...

$$\begin{array}{rr} & A + B = 45 \\ \text{SUBTRACT} & -8A + B = 18 \\ \hline & 9A = 27 \end{array}$$

So $A = 3$ and $B = 42$

The particular solution is,

$$y = 3e^{-8x} + 42e^{x} - 9$$

[8 marks]

Answer 2

$$y'' + 12y' + 37y = 10\,e^{-4x}$$

Obtain the complementary function by solving the homogeneous equation,

$$y'' + 12y' + 37y = 0$$

$$m^2 + 12m + 37 = 0 \quad \text{is the auxiliary equation}$$

$$m = \frac{-12 \pm \sqrt{144 - 4 \times 37}}{2}$$

$$= \frac{-12 \pm \sqrt{(-1)(4)}}{2}$$

$$= -6 \pm i \quad \text{(complex roots)}$$

$\therefore$ The complementary function is $y = e^{-6x}(A\cos x + B\sin x)$

Particular integral: $y = Ue^{-4x}$ so $y' = -4Ue^{-4x}$ and $y'' = 16Ue^{-4x}$

Substitute these into the question's differential equation...

$$16Ue^{-4x} - 48Ue^{-4x} + 37Ue^{-4x} = 10\,e^{-4x}$$

$$5U = 10$$

$$U = 2$$

The particular integral is $y = 2\,e^{-4x}$

The general solution is $y = e^{-6x}(A\cos x + B\sin x) + 2\,e^{-4x}$

from which $y' = e^{-6x}(-A\sin x + B\cos x)$

$$- 6\,e^{-6x}(A\cos x + B\sin x) - 8\,e^{-4x}$$

From the boundary conditions $y'(0) = 0,\ y(0) = 4$

$$B - 6A - 8 = 0 \text{ and } A + 2 = 4$$

So $A = 2$ and $B = 20$

The particular solution is,

$$y = e^{-6x}(2\cos x + 20\sin x) + 2\,e^{-4x}$$

[9 marks]

Answer 3

$$\frac{d^2y}{dx^2} + k^2 y = 0, \ y = 1 \text{ and } \frac{dy}{dx} = 1 \text{ at } x = 0$$

$$m^2 + k^2 = 0 \text{ is the auxiliary equation}$$

$$m^2 - (k\mathrm{i})^2 = 0$$

$$m^2 = (k\mathrm{i})^2$$

$$m = \pm k\mathrm{i} \qquad \text{(complex roots)}$$

$\therefore$ The complementary function is $y = A\cos kx + B\sin kx$

Which is also the general solution to the homogeneous differential equation.

$$\frac{dy}{dx} = -Ak\sin kx + Bk\cos kx$$

$$\frac{d^2y}{dx^2} = -Ak^2\cos kx - Bk^2\sin kx$$

Substitute these into the question's differential equation...

$$-Ak^2\cos kx - Bk^2\sin kx + k^2(A\cos kx + B\sin kx) = 0$$

which is consistent, as it should be

Making use of the boundary conditions,

$$y = A\cos kx + B\sin kx \Rightarrow 1 = A$$

$$\frac{dy}{dx} = -Ak\sin kx + Bk\cos kx \Rightarrow 1 = Bk \text{ so } B = \frac{1}{k}$$

$$\therefore \ y = \cos kx + \frac{1}{k}\sin kx$$

[8 marks]

Answer 4

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 27\,e^{-x}$$

(a) Obtain the complementary function by solving the homogeneous equation,

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 0$$

$$m^2 + 2m + 10 = 0 \quad \text{is the auxiliary equation}$$

$$m = \frac{-2 \pm \sqrt{4 - 4 \times 1 \times 10}}{2}$$

$$= \frac{-2 \pm \sqrt{(-1)(36)}}{2}$$

$$= -1 \pm 3\mathrm{i} \quad \text{(complex roots)}$$

$\therefore$ The complementary function is $y = e^{-x}(A\cos 3x + B\sin 3x)$

For a particular integral, $y = Ue^{-x}$

$$\frac{dy}{dx} = -Ue^{-x} \text{ and } \frac{d^2y}{dx^2} = Ue^{-x}$$

Substitute these into the question's differential equation...

$$Ue^{-x} - 2Ue^{-x} + 10Ue^{-x} = 27e^{-x}$$

$$9U = 27$$

$$U = 3$$

$\therefore$ The particular integral is $y = 3\,e^{-x}$

The general solution is $y = e^{-x}(A\cos 3x + B\sin 3x + 3)$

[6 marks]

(b) $\dfrac{dy}{dx} = e^{-x}(-3A\sin 3x + 3B\cos 3x) - e^{-x}(A\cos 3x + B\sin 3x + 3)$

Making use of the boundary conditions, $y = 0$ and $\dfrac{dy}{dx} = 0$ when $x = 0$

$$0 = A + 3 \Rightarrow A = -3$$

$$0 = 3B - A - 3 \Rightarrow B = 0$$

Particular solution: $y = 3e^{-x}(1 - \cos 3x)$

[6 marks]

Answer 5

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 26 \sin 3x$$

(a) Obtain the complementary function by solving the homogeneous equation,

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} = 0$$

$$m^2 - 2m = 0 \text{ is the auxiliary equation}$$

$$m(m - 2) = 0$$

Either $m = 0$ or $m = 2$ (two distinct real roots)

$\therefore$ The complementary function is $y = A + Be^{2x}$

For a particular integral try $y = U\cos 3x + V\sin 3x$

in which case $\dfrac{dy}{dx} = -3U\sin 3x + 3V\cos 3x$

and $\dfrac{d^2y}{dx^2} = -9U\cos 3x - 9V\sin 3x$

Substitute these into the question's differential equation...

$$-9U\cos 3x - 9V\sin 3x + 6U\sin 3x - 6V\cos 3x = 26 \sin 3x$$

Matching coefficients of $\cos 3x$: $-9U - 6V = 0$

$$-3U - 2V = 0$$

Matching coefficients of $\sin 3x$: $6U - 9V = 26$

$$-6U - 4V = 0$$

$$ADD \qquad 6U - 9V = 26$$

$$-13V = 26$$

$$V = -2 \text{ and } U = \frac{4}{3}$$

$\therefore$ The particular integral is $y = \dfrac{4}{3}\cos 3x - 2\sin 3x$

The general solution is $y = A + Be^{2x} + \dfrac{4}{3}\cos 3x - 2\sin 3x$

[8 marks]

(b) $$\frac{dy}{dx} = 2Be^{2x} - 4\,sin\,3x - 6\,cos\,3x$$

Using the boundary conditions that $y = 0$ and $\frac{dy}{dx} = 0$ when $x = 0$

$$y = A + Be^{2x} + \frac{4}{3}cos\,3x - 2\,sin\,3x \Rightarrow 0 = A + B + \frac{4}{3}$$

$$\frac{dy}{dx} = 2Be^{2x} - 4\,sin\,3x - 6\,cos\,3x \quad \Rightarrow 0 = 2B - 6$$

So, $B = 3$ and $A = -\frac{13}{3}$

The particular solution is, $y = -\frac{13}{3} + 3e^{2x} + \frac{4}{3}cos\,3x - 2\,sin\,3x$

[5 marks]