GCSE Mathematics

ITERATION

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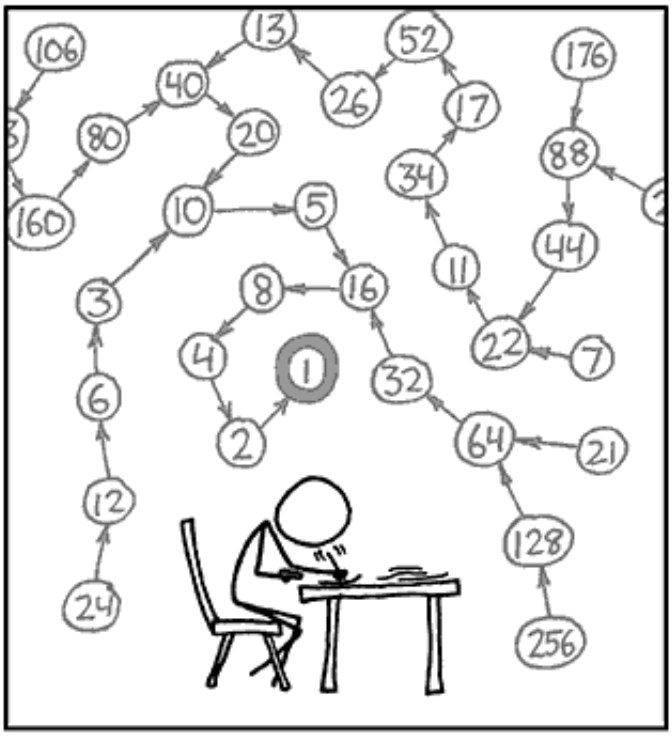


THE COLLATZ CONJECTURE STATES THAT IF YOU PICK A NUMBER, AND IF IT'S EVEN DIVIDE IT BY TWO AND IF IT'S ODD MULTIPLY IT BY THREE AND ADD ONE, AND YOU REPEAT THIS PROCEDURE LONG ENOUGH, EVENTUALLY YOUR FRIENDS WILL STOP CALLING TO SEE IF YOU WANT TO HANG OUT.

I T E R A T I O N

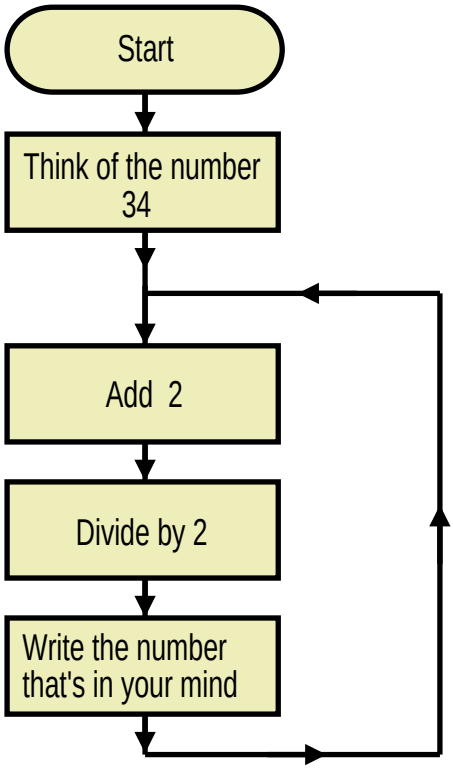
Lesson 1

1.1 What is a Iteration ?

An iterative process is one in which you repeatedly carry out the same set of instructions. The idea of iteration has been around for over two hundred years but it is only since the invention of the desktop computer in the 1980s that the subject has become mainstream mathematics. A flowchart provides a convenient way of describing an iteration.

1.2 Example

Consider the following flowchart,



This flowchart generates a sequence of numbers.
We can give the sequence a name; sequence *U*.
The first term in sequence *U* is denoted U_1
The second term in sequence *U* is denoted U_2
And so on...
Complete this table to show the first eight terms in sequence *U*

U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8
34							

[6 marks]

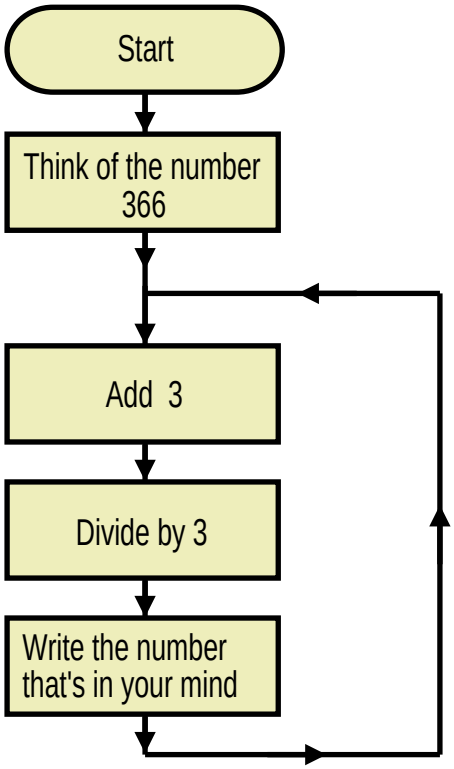
1.3 Exercise

Non-Calculator

Marks Available : 40

Question 1

Consider the following flowchart,



This flowchart generates a sequence of numbers.
We can give the sequence a name; sequence V
The first term in sequence V is denoted V_1
The second term in sequence V is denoted V_2
And so on...
Complete this table to show the first seven terms in sequence V

V_1	V_2	V_3	V_4	V_5	V_6	V_7
366						

[6 marks]

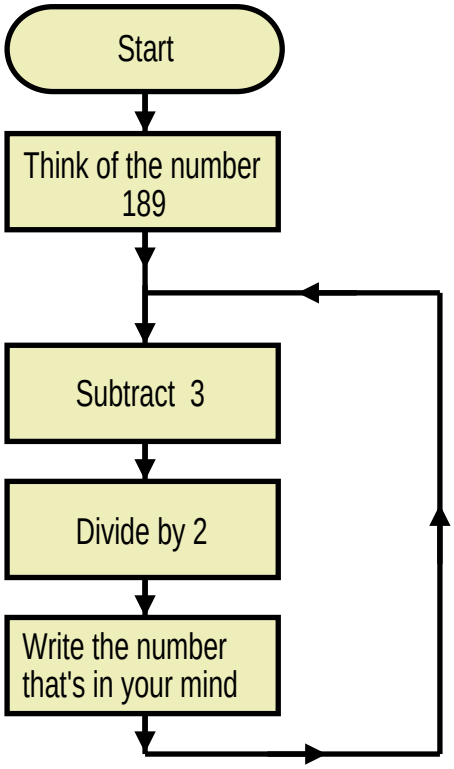
Question 2

This question will tell you how to get terms in sequence Q.
Start with the number 1, that is $Q_1 = 1$
To get a next term, double the previous term.
What is Q_6 ?

[2 marks]

Question 3

Consider the following flowchart,



This flowchart generates the sequence of numbers *D*.

Complete this table to show the first seven terms in sequence *D*

D_1	D_2	D_3	D_4	D_5	D_6	D_7
189						

[6 marks]

Question 4

This question will tell you how to get terms in sequence *A*.

Start with the number one million, that is $A_1 = 1000000$

To get a next term, divide the previous term by 10.

(i) What is the value of A_2 ?

[1 mark]

(ii) What is the value of A_3 ?

[1 mark]

(iii) What is the value of A_7 ?

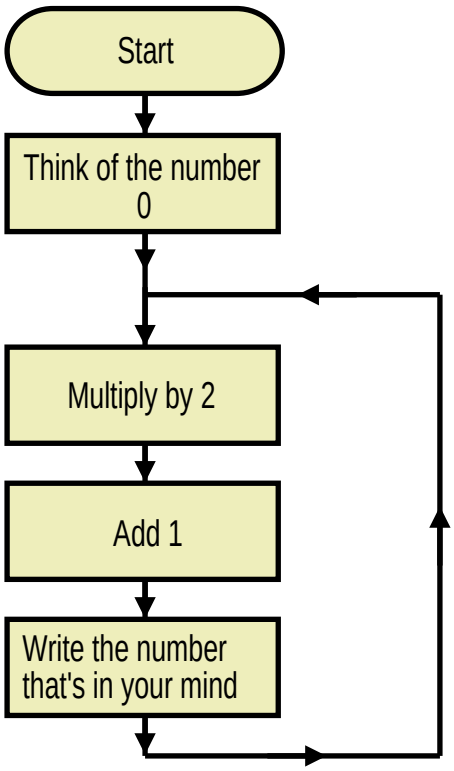
[1 mark]

(iv) What is the value of A_9 ?

[2 marks]

Question 5

Consider the following flowchart,



This flowchart generates the sequence of numbers P

Complete this table to show the first ten terms in sequence P

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}
			7						

[6 marks]

Question 6

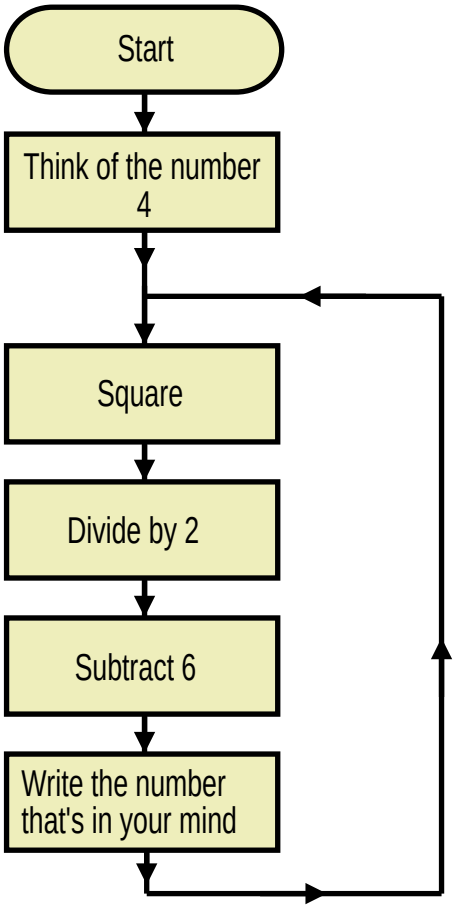
In my garden on Monday at noon there are 100 snails.

At noon each day, there are 20% more snails than at noon the day before.

How many snails are in my garden at noon on Wednesday that week ?

[3 marks]

Question 7



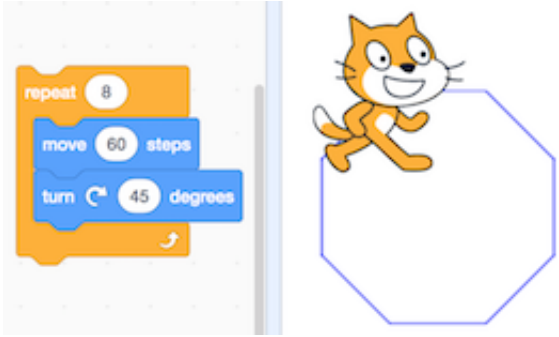
This flowchart which behaves in a different manner to the others in this exercise.
Call the sequence of numbers generated by this flowchart *Z*
Complete this table to show the first ten terms in sequence *Z*

Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7	Z_8	Z_9	Z_{10}

[6 marks]

Question 8

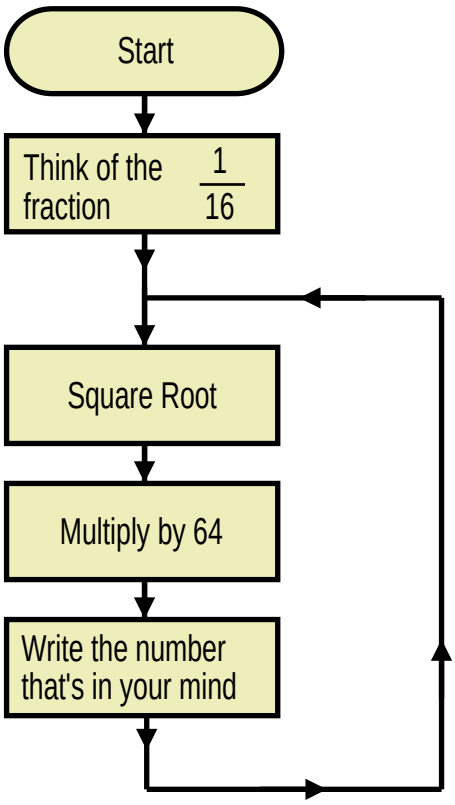
Explain how to modify this “scratch” programming loop to draw a hexagon.



[2 marks]

Question 9

See if you can you handle this question without using a calculator.
(It's all to do with powers of 2)



This flowchart generates the sequence of numbers C
Complete this table to show the first five terms in sequence C

P_1	P_2	P_3	P_4	P_5
$\frac{1}{16}$				

[4 marks]

1.4 Answers

GCSE Mathematics
Iteration

1.4.1 Teaching Notes

With a high ability class you may like to ask for the mathematical iterative description of each flowchart and these are included in the answers below. More routinely, the iterative descriptions can be shown when going over the answers. For pupils that finish early, they could be asked to get the *scratch* program of question 8 running on their laptop or other device. This is via the interactive webpage at https://scratch.mit.edu/projects/editor/?tutorial=getStarted

1.4.2 Solution to 1.2 Example

$$U_1 = 34, \qquad U_{n+1} = \frac{U_n + 2}{2}$$

U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8
34	18	10	6	4	3	2.5	2.25

[6 marks]

1.4.3 Solutions to 1.3 Exercise

Answer 1

$$V_1 = 366, \qquad V_{n+1} = \frac{V_n + 3}{3}$$

V_1	V_2	V_3	V_4	V_5	V_6	V_7
366	123	42	15	6	3	2

[6 marks]

Answer 2

$$Q_1 = 1, \qquad Q_{n+1} = 2\,Q_n$$

Q_1	Q_2	Q_3	Q_4	Q_5	Q_6
1	2	4	8	16	32

$$Q_6 = 32$$

[2 marks]

Answer 3

$$D_1 = 189, \qquad D_{n+1} = \frac{D_n - 3}{2}$$

D_1	D_2	D_3	D_4	D_5	D_6	D_7
189	93	45	21	9	3	0

[6 marks]

Answer 4

$$A_1 = 1000000, \qquad A_{n+1} = \frac{A_n}{10}$$

(i) $A_2 = 100000$

[1 mark]

(ii) $A_3 = 10000$

[1 mark]

(iii) $A_7 = 1$

[1 mark]

(iv) $A_9 = 0.01$

[2 marks]

Answer 5

$$P_1 = 0, \qquad P_{n+1} = 2P_n + 1$$

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}
0	1	3	7	15	31	63	127	255	511

[6 marks]

Answer 6

$$S_1 = 100, \qquad S_{n+1} = 1.2\,S_n$$

$$S_2 = 120$$

$S_3 = 144$ snails at noon on Wednesday

[3 marks]

Answer 7

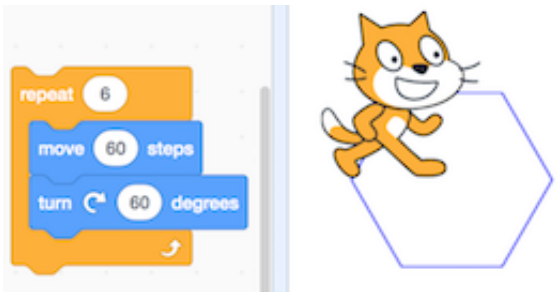
$$X_1 = 4, \qquad X_{n+1} = \frac{(X_n)^2}{2} - 6$$

Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7	Z_8	Z_9	Z_{10}
4	2	– 4	2	– 4	2	– 4	2	– 4	2

After the initial term the iteration oscillates between the value 2 and the value – 4

[6 marks]

Answer 8



[2 marks]

Answer 9

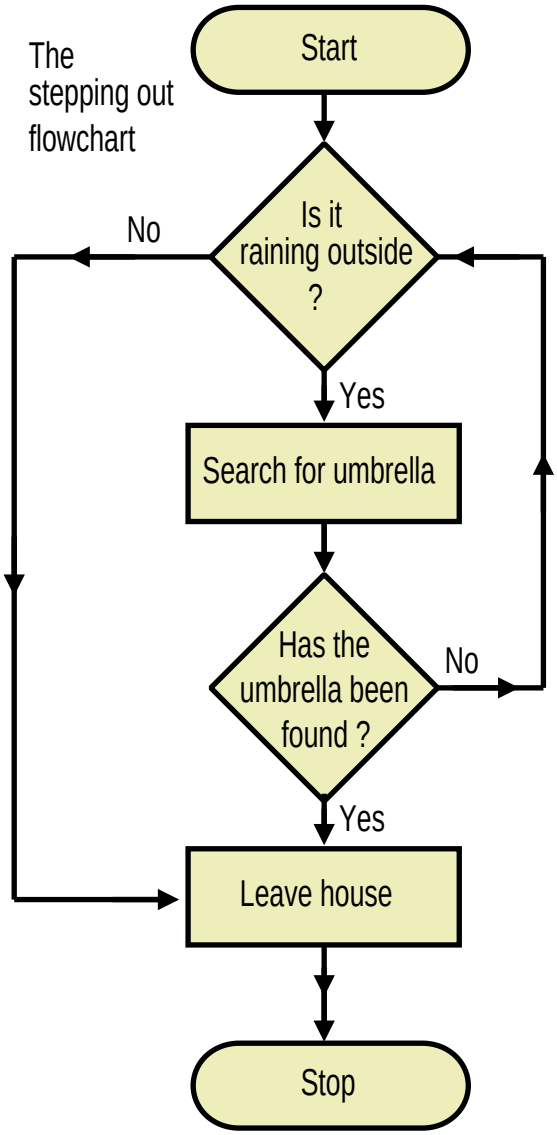
$$U_1 = \frac{1}{16} \qquad U_{n+1} = 64\sqrt{U_n}$$

P_1	P_2	P_3	P_4	P_5
$\frac{1}{16}$	16	256	1024	2048

[4 marks]

2.1 Decisions, Decisions

The following flowchart includes an extra feature in the form of a couple of decision diamonds. When a decision is to be made only two answers to the question are allowed; one of either "yes" or "no".



(i) What happens if it keeps raining and the umbrella cannot be found ?

[1 mark]

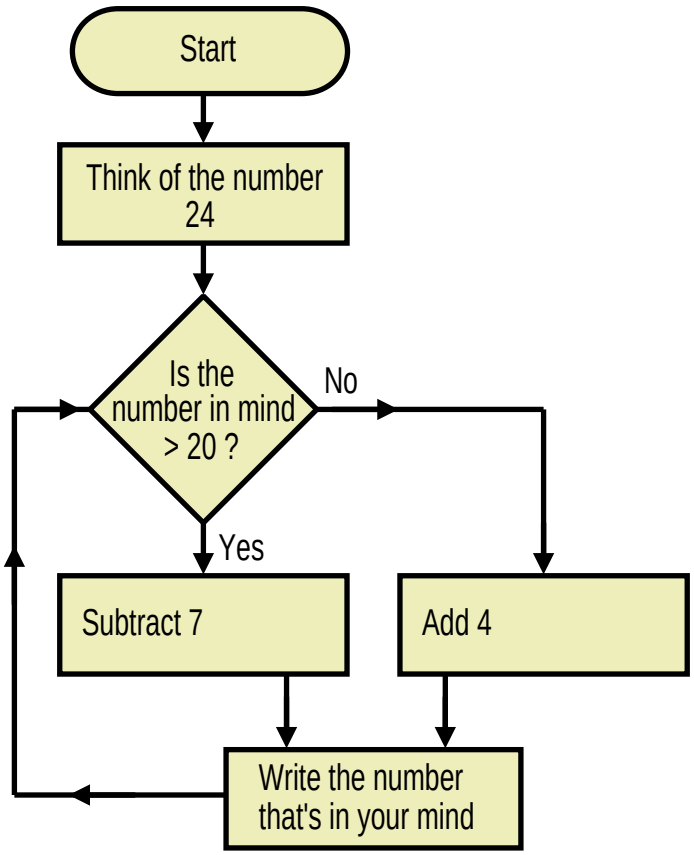
(ii) Is it possible to leave the house when it is not raining but with the umbrella ? Explain your answer.

[2 marks]

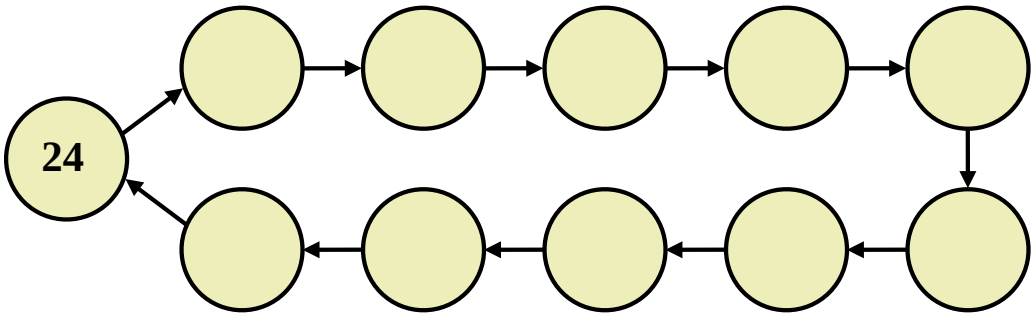
(iii) How might the issues raised by parts (i) and (ii) be resolved ?

[2 marks]

2.2 Example



The flowchart generates a loop of numbers.
On the following diagram write out the numbers that are in the loop.



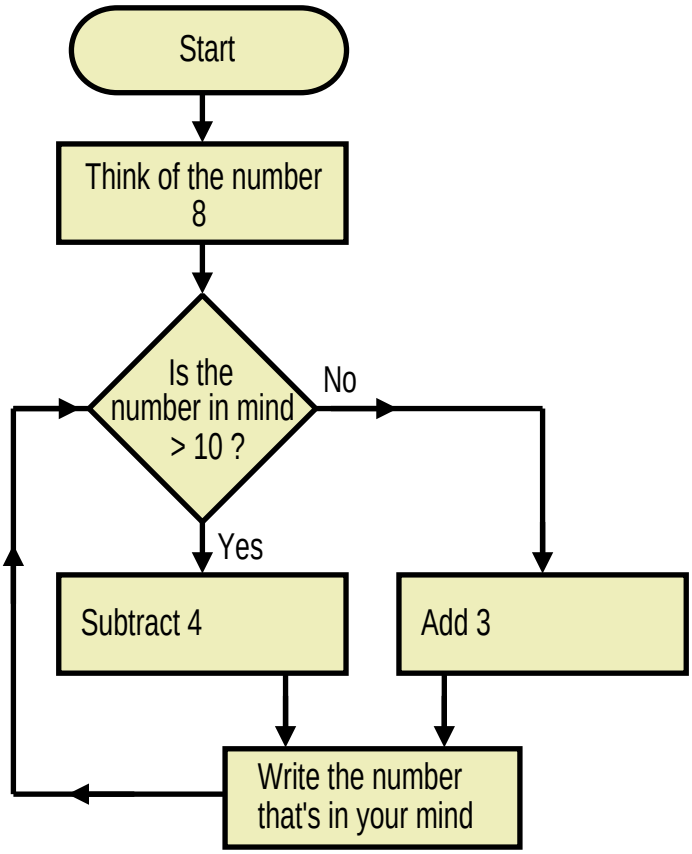
[6 marks]

2.3 Exercise

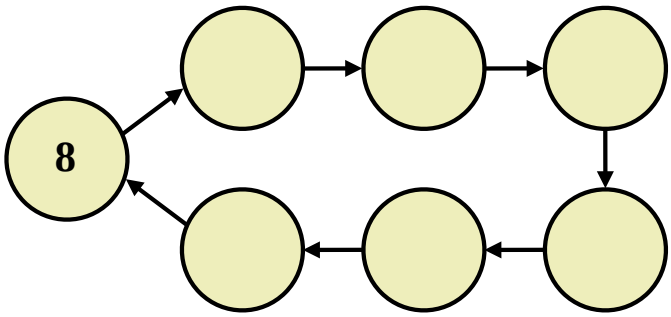
Non-Calculator

Marks Available : 40

Question 1

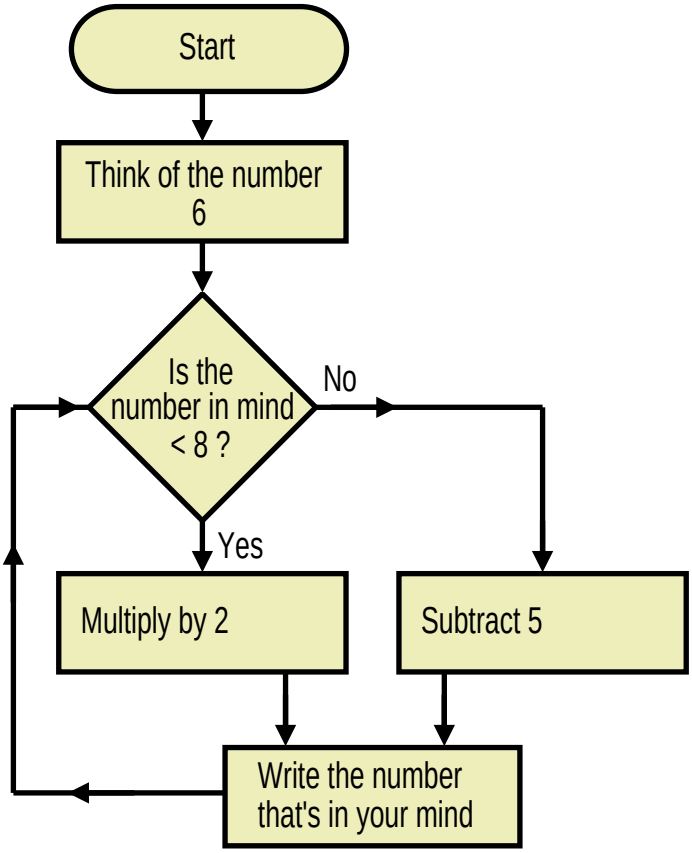


The following flowchart generates a loop of numbers.
On the following diagram write out the numbers that are in the loop.

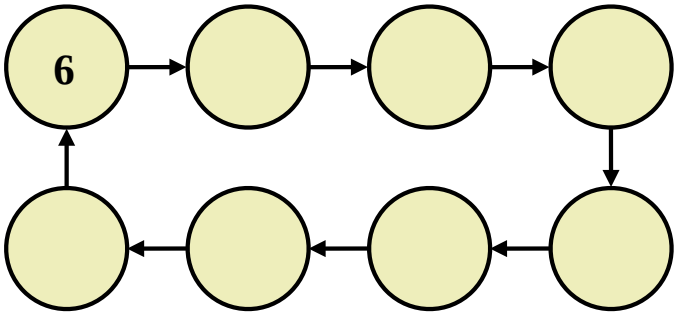


[6 marks]

Question 2

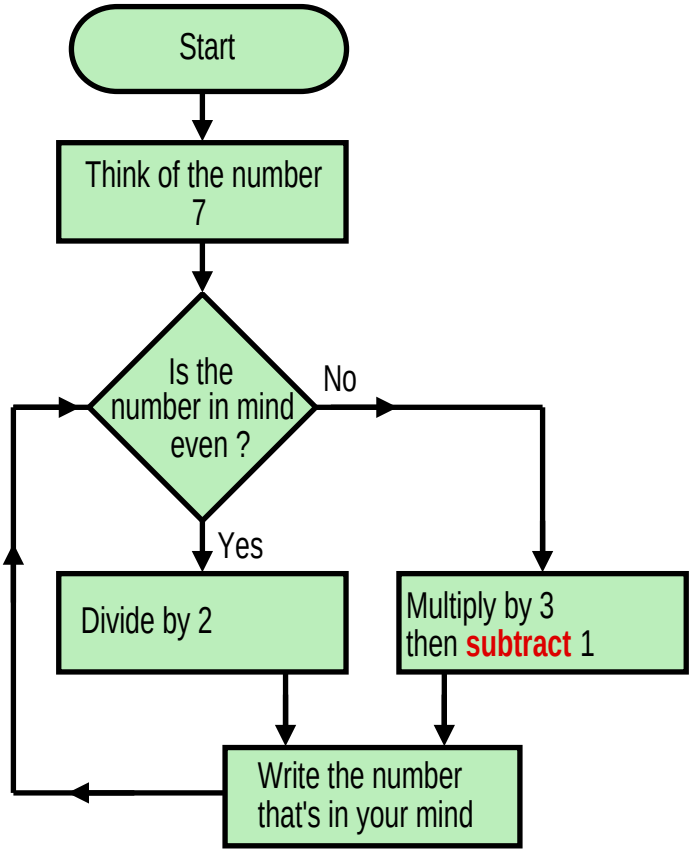


The following flowchart generates a loop of numbers.
On the following diagram write out the numbers that are in the loop.

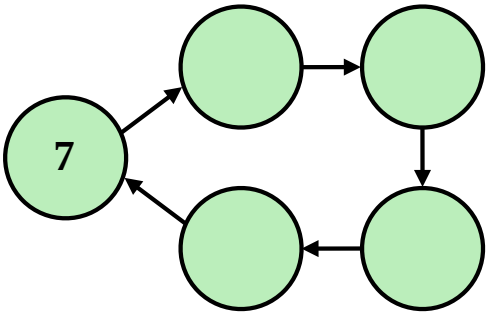


[6 marks]

Question 3



The following flowchart generates a loop of numbers.
On the following diagram write out the numbers that are in the loop.

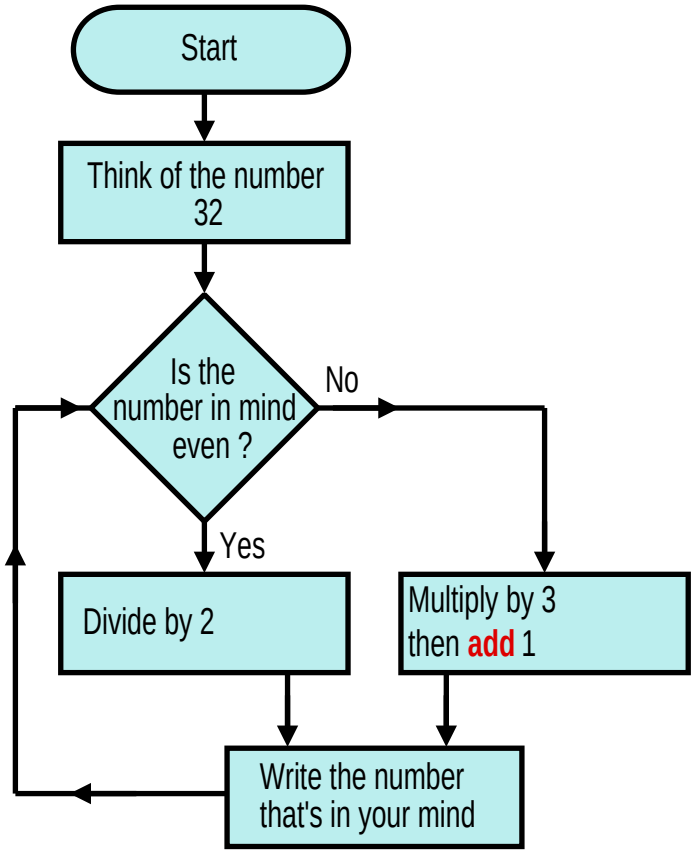


[6 marks]

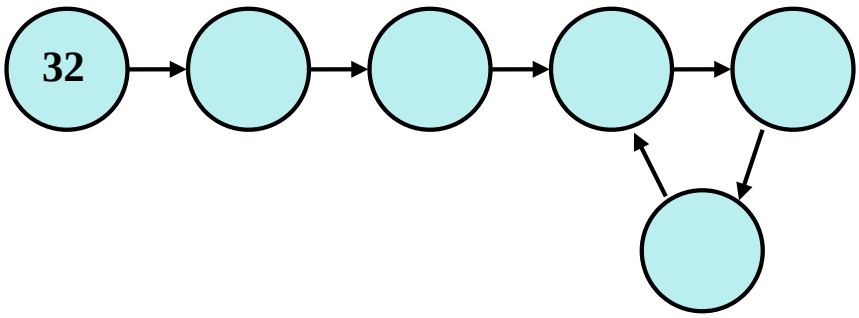
Question 4

(i)

Now to look at a problem that no-one has yet been able to solve since it was first stated in 1937 called the Collatz conjecture. It's about an iteration that the conjecture claims will always end up going around in a loop that includes the number 1. It will do this, so it is claimed, no matter what starting number is used. Here is the flowchart for the iteration with a starting number of 32.



On the following diagram write out the numbers that are generated by the flowchart.

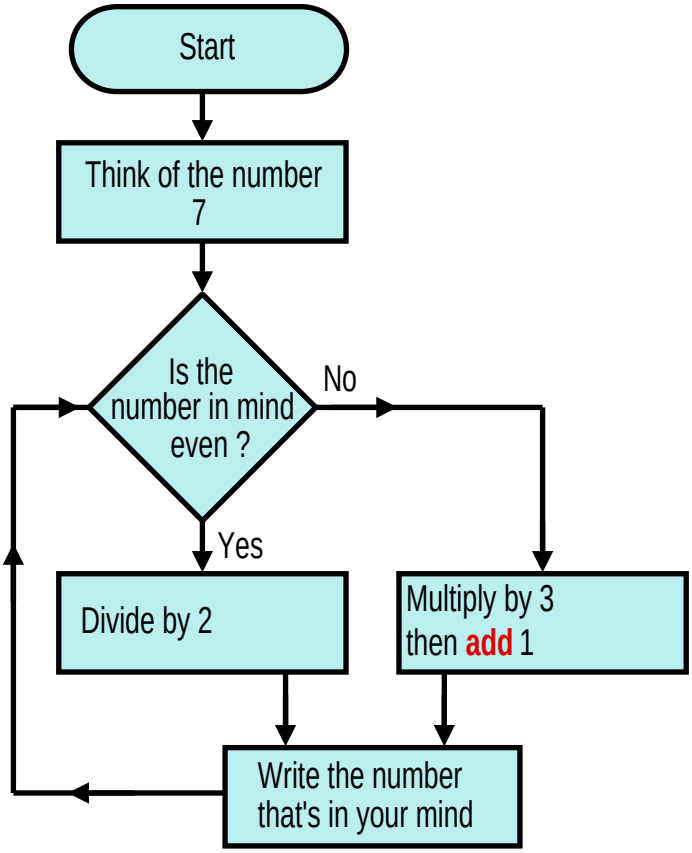


This is an example of a starting number that “falls to earth”.
Not all numbers do this right away as you will discover next.

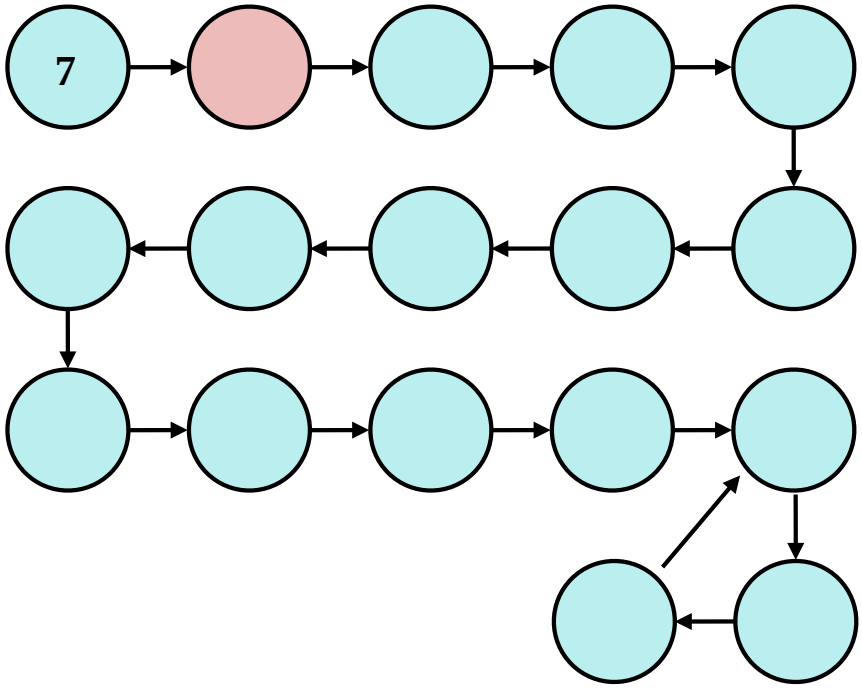
[4 marks]

(ii)

Here is the flowchart for the same iteration but starting with the number of 7.



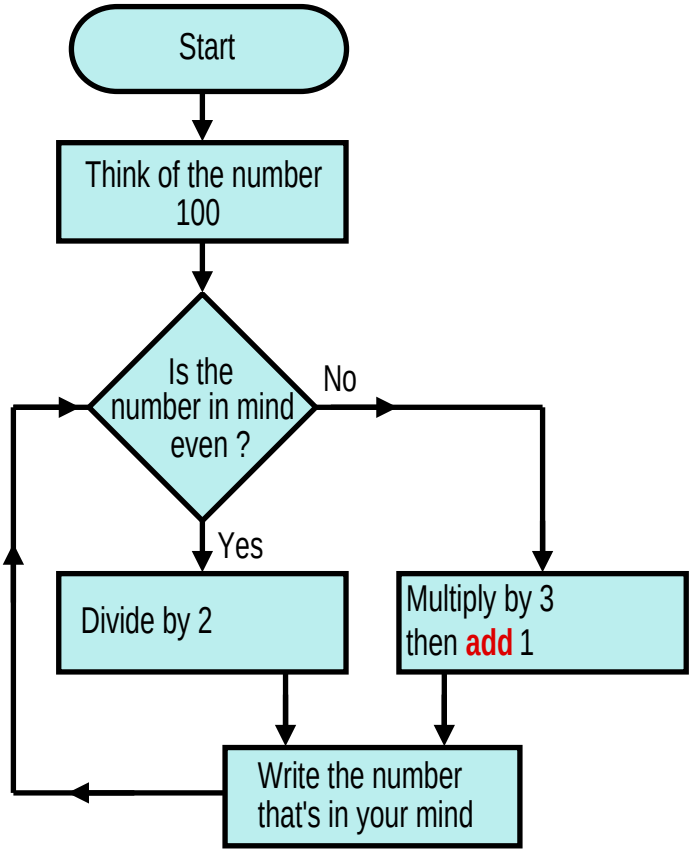
On the following diagram write out the numbers that are generated.

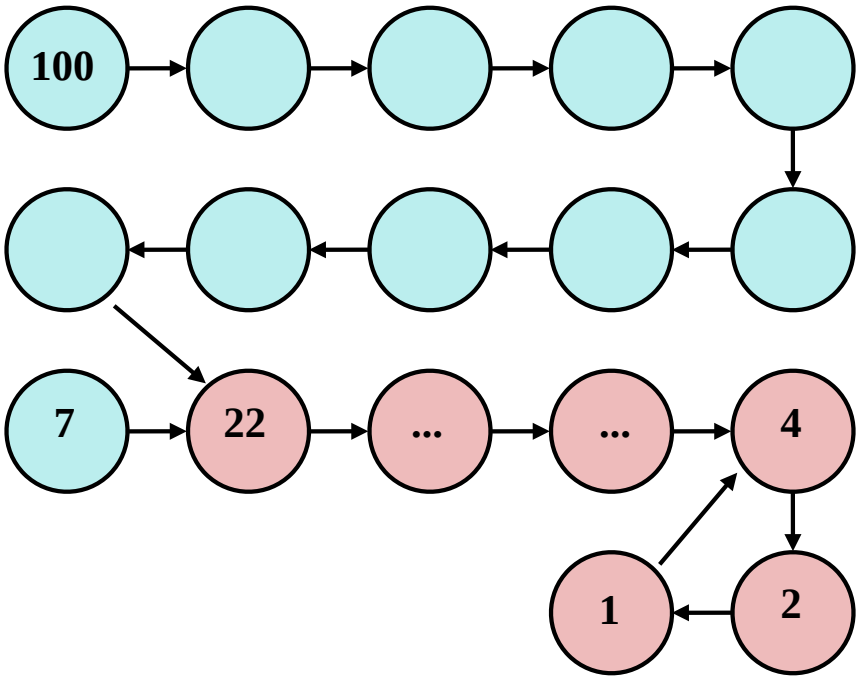


[8 marks]

(iii)

From part (iii) you know what happens to many other starting numbers. For example, in the red circle you will have the number 22 which means that you know what happens if 22 is used as the start number. So, the only numbers worth trying now are numbers that did not occur in any of the circles of your part (i) or part (ii) answers. Let's try 100 next. Remember: No calculator !

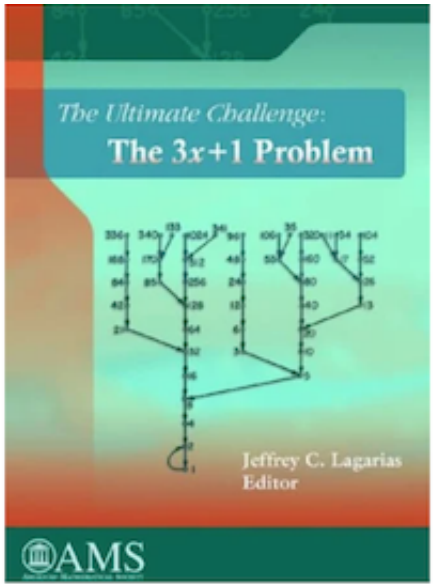




[8 marks]

(iv)

Entire books have been written about the Collatz conjecture. To prove the conjecture is not true you would only have to find a starting number for which the loop at the end does not include the number one. No-one has yet found such a loop but people are still searching because nobody has managed to prove that the end loop will always include the number one. If you could prove that the Collatz conjecture is true you would be on the front pages of the world's newspapers. Similarly if you found a starting number that showed it is false.



The Ultimate Challenge: The 3x+1 Problem [Book] £54.50

Hardback · Non-fiction Amazon.co.uk

Watch the following Numberphile video in which Professor David Eisenbud talks about the Collatz Conjecture (about 8 minutes).

Teaching Video : https://www.NumberWonder.co.uk/v9120/2.mp4

[2 mark]

2.4 Answers

2.4.1 Solution to 2.1 Discussion

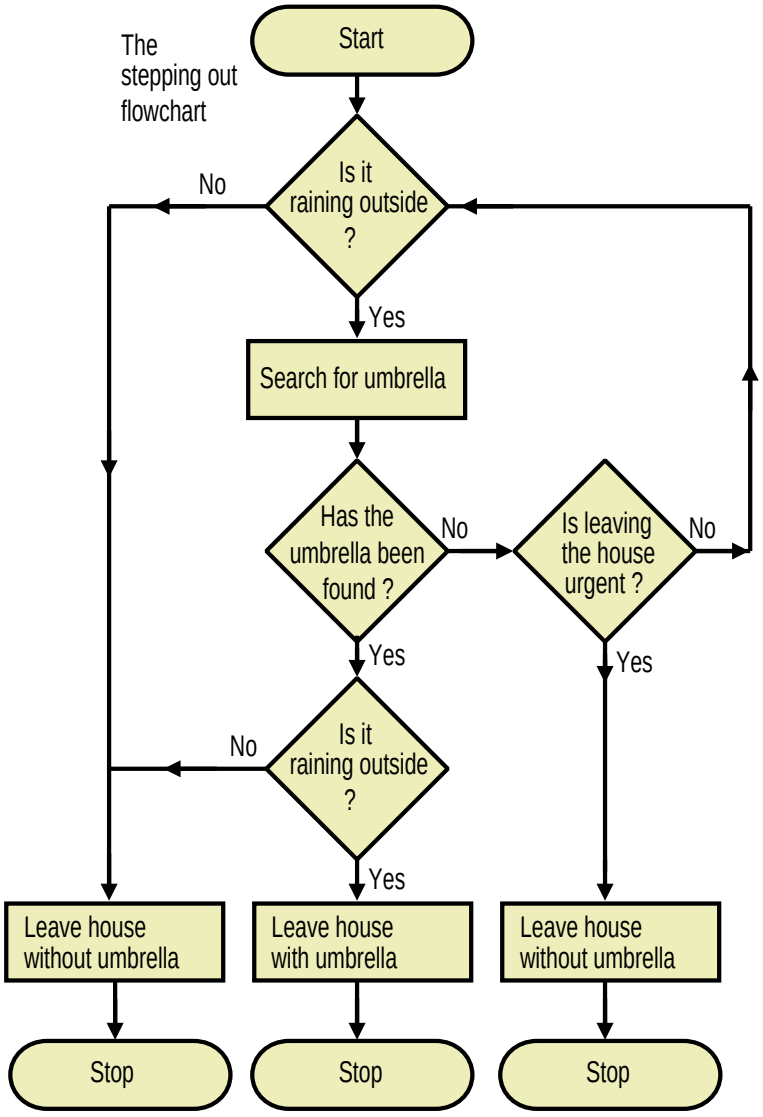
(i) If the umbrella cannot be found the searcher cannot leave the house until it stops raining. They are “stuck in a loop”.

[1 mark]

(ii) YES - It is possible to leave the house when it is not raining but with the umbrella. This would happen if it was raining but then stopped whilst searching for the umbrella, the umbrella then being found.

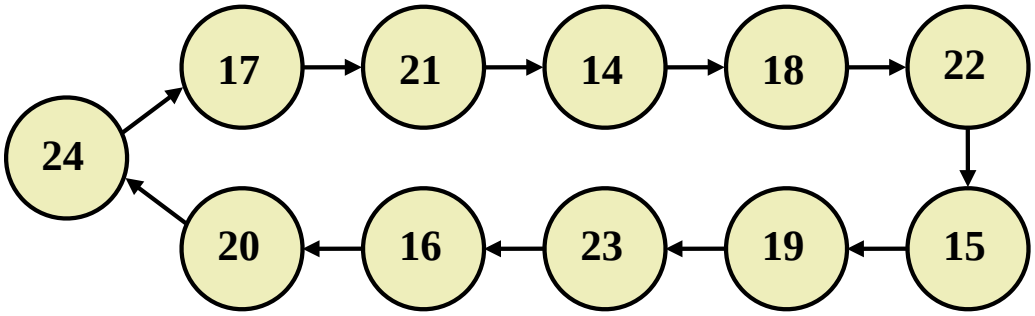
[2 marks]

(iii) A more complicated flowchart could be devised to prevent the situations in parts (i) and (ii) arising by adding more decision diamonds (but you may feel that leaving the house with an umbrella when it's only just stopped raining is a wanted result anyway).



[2 marks]

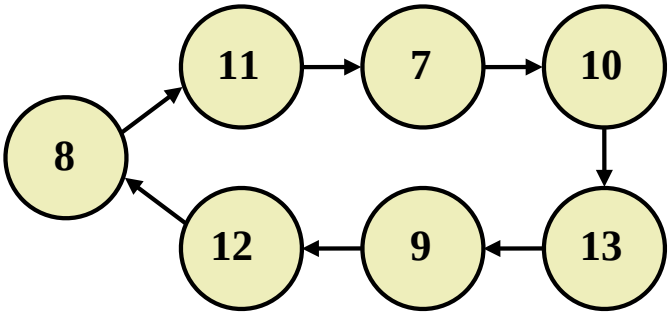
2.4.2 Solution to 2.2 Example



[6 marks]

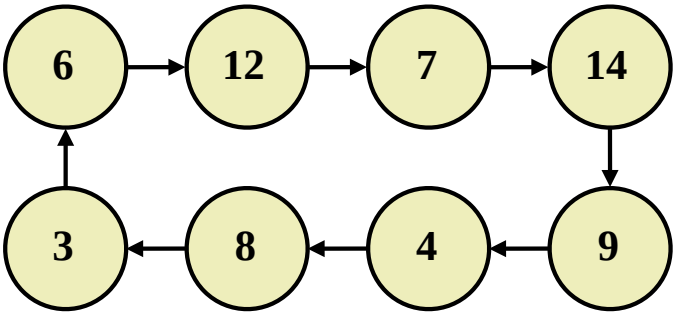
2.4.3 Solution to 2.3 Exercise

Answer 1



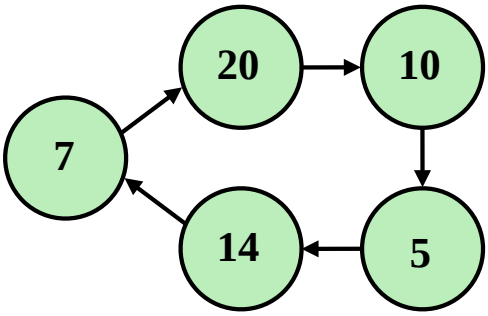
[6 marks]

Answer 2



[6 marks]

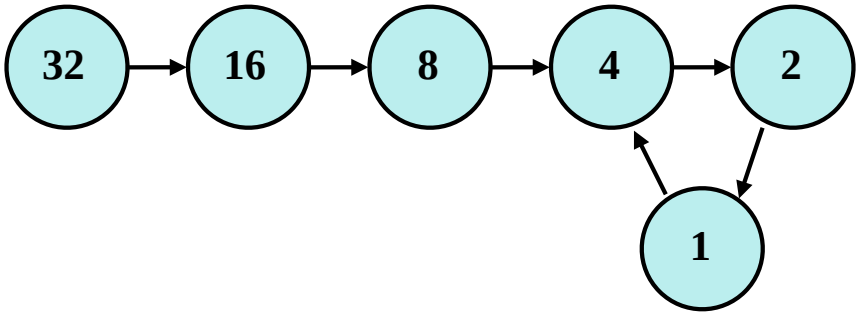
Answer 3



[6 marks]

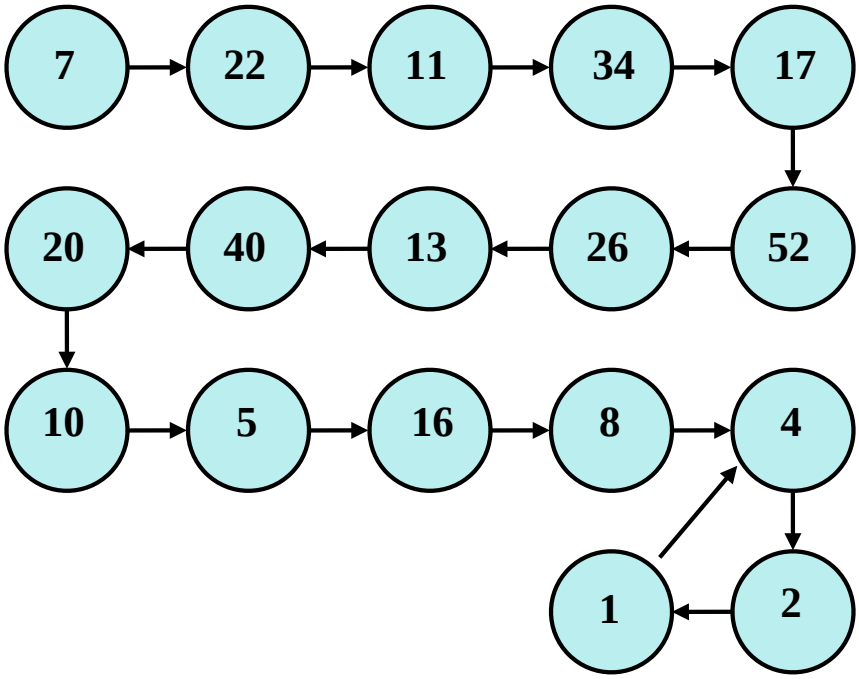
Answer 4

(i)



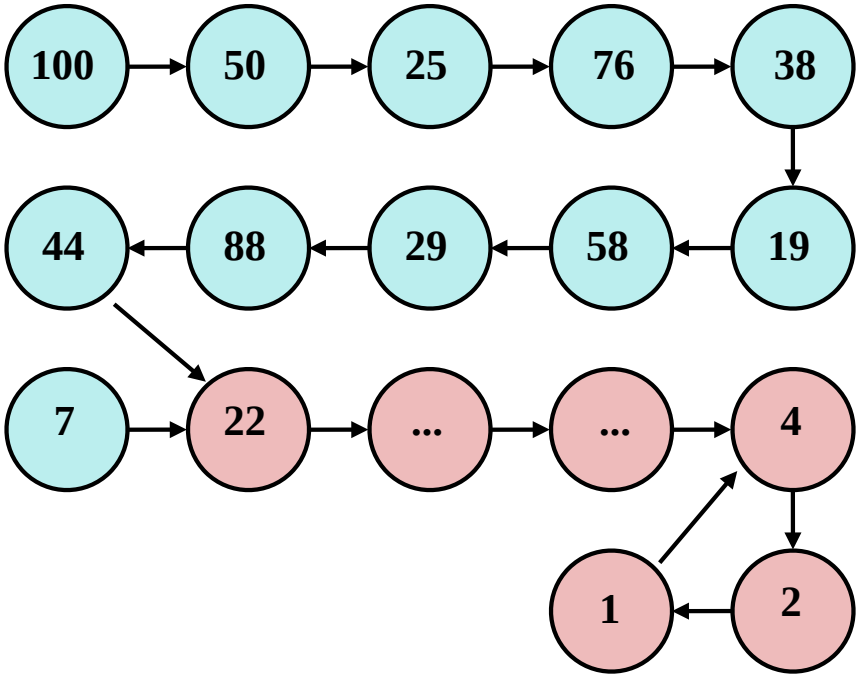
[4 marks]

(ii)



[8 marks]

(iii)



[8 marks]

(iv) Watch video !

[2 marks]

Lesson 3

GCSE Mathematics
Iteration

3.1 Coding The Flowchart

Although a flowchart is an easily understood way of describing an iterative process it takes up a lot of space on the page and is time consuming to draw. Mathematicians have devised a much more clever way of describing an iteration. Here's how they translate two key phrases about a sequence *U* into mathematics,

"the term of focus" becomes U_n

"the next term (one step on from the term of focus)" becomes U_{n+1}

Complete the following table;

n	U_n	U_{n+1}
1	U_1	$U_{1+1} = U_2$
2		
3		
...		
8		
...		
43		

[3 marks]

3.2 Example

A number sequence, *U*, has the following iterative description,

$$U_1 = 0 \qquad U_{n+1} = 3U_n + 1$$

Note the difference in meaning between $n + 1$ in a small font size with that in larger.
Complete the table to show the first eight terms of the sequence.

U_1	U_2	U_3	U_4	U_5	U_6	U_7

[4 marks]

3.3 Exercise

Non-Calculator

Marks Available : 50

Question 1

A number sequence, A, has the following iterative description,

$$A_1 = 2 \qquad A_{n+1} = 2A_n - 1$$

Complete the table to show the first seven terms of the sequence.

A_1	A_2	A_3	A_4	A_5	A_6	A_7

[4 marks]

Question 2

A number sequence, H, has the following iterative description,

$$H_1 = 11 \qquad H_{n+1} = 2A_n - 10$$

Complete the table to show the first eight terms of the sequence.

H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8

[4 marks]

Question 3

Professor RE Peat believes that the following iteration will always generate a prime number.

$$P_1 = 5 \qquad P_{n+1} = 2P_n - 3$$

(i) Complete the table to show the first six terms of the sequence.

P_1	P_2	P_3	P_4	P_5	P_6

[4 marks]

(ii) Considering just the six terms from part (i), might the professor be correct ? Give a reason for your answer.

[2 marks]

Question 4

A number sequence, *B*, has the following iterative description,

$$B_1 = 10 \qquad B_{n+1} = 3B_n - 10$$

Complete the table to show the first seven terms of the sequence.

B_1	B_2	B_3	B_4	B_5	B_6	B_7

[4 marks]

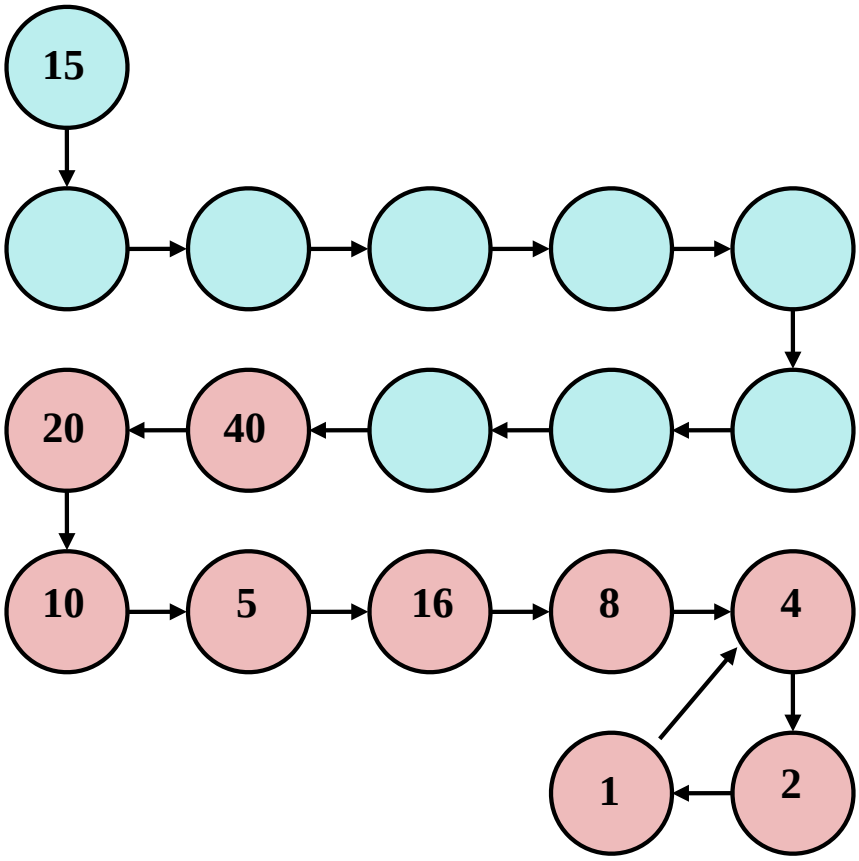
Question 5

Last lesson we looked at an unsolved mathematics problem; the Collatz conjecture. It had a flowchart that did a different calculation for the next term in the sequence depending upon if the "number in mind" was even or odd.

Here is how that flowchart is described mathematically,

$$C_{n+1} = \begin{cases} \frac{C_n}{2} & \text{if } C_n \text{ is even} \\ 3C_n + 1 & \text{if } C_n \text{ is odd} \end{cases}$$

Starting with 15, work out the 8 missing terms before this new branch of tree joins onto the main trunk (worked out last lesson).



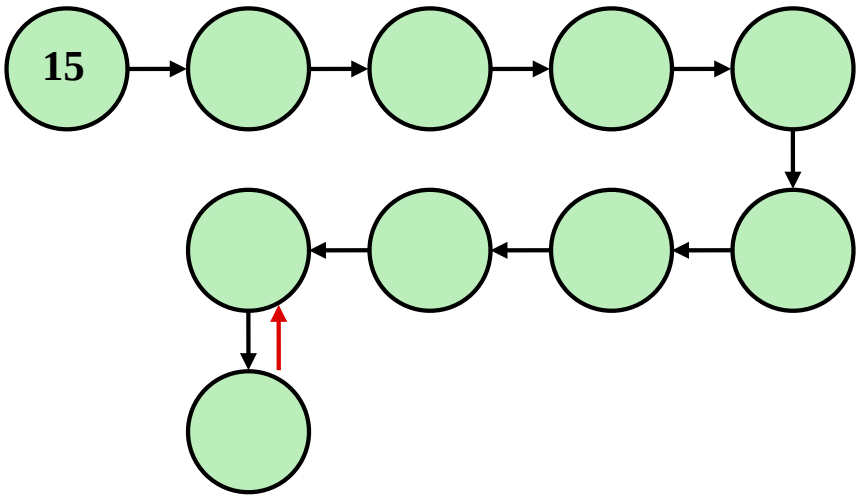
[6 marks]

Question 6

It's natural to wonder if the Collatz conjecture holds for other similar iterations. Consider this very similar rule where the “add 1” is changed to “subtract 1”.

$$L_{n+1} = \begin{cases} \frac{L_n}{2} & \text{if } L_n \text{ is even} \\ 3L_n - 1 & \text{if } L_n \text{ is odd} \end{cases}$$

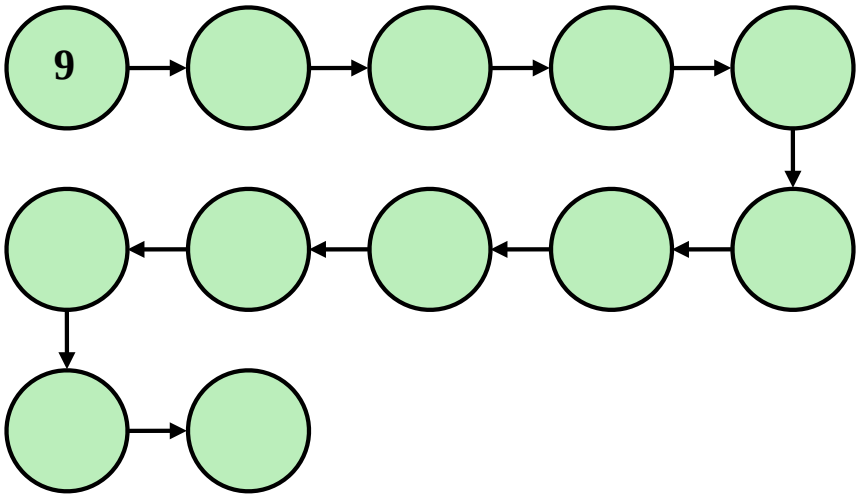
(i) On the following diagram write out the numbers generated.if $L_1 = 15$



This suggests that the new rule may be behaving much like the old.

[6 marks]

(ii) On the following diagram write out the numbers generated.if $L_1 = 9$



[6 marks]

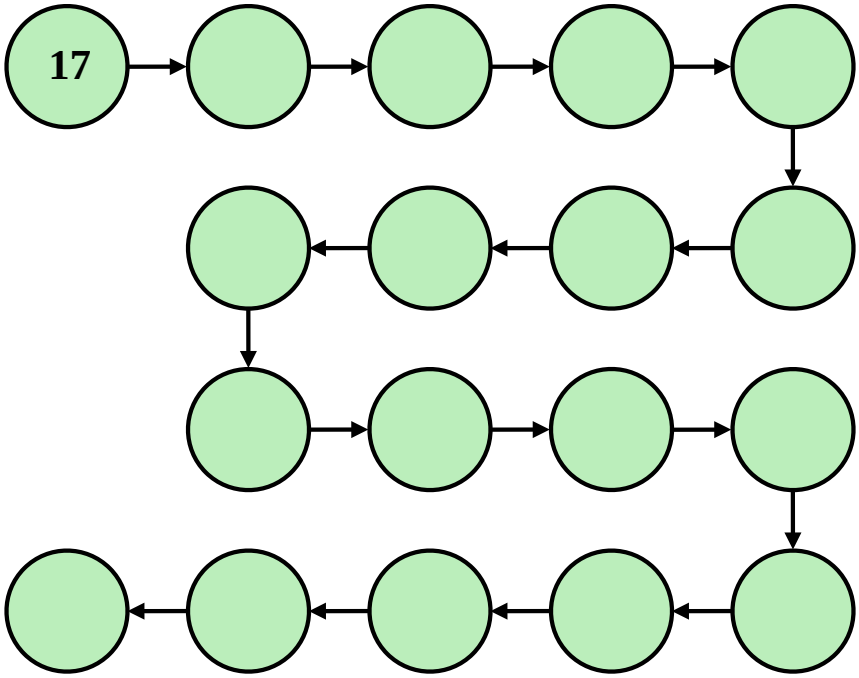
(iii) Add an arrow to the last circle of your part (ii) answer to show how it connects back to a previous circle in the sequence.

[1 mark]

(iv) Explain how your part (ii) and (iii) answers show that the Collatz conjecture is FALSE for the adjusted rule.

[2 marks]

(v) On the following diagram write out the numbers generated.if $L_1 = 17$



[10 marks]

(vi) Add an arrow to the last circle of your part (v) answer to show how it connects back to a previous circle in the sequence.

[1 mark]

3.4 Answers

3.4.1 Solution to 3.1 Table

n	U_n	U_{n+1}
1	U_1	$U_{1+1} = U_2$
2	U_2	U_3
3	U_3	U_4
...	...	...
8	U_8	U_9
...	...	...
43	U_{43}	U_{44}

[3 marks]

3.4.2 Solution to 3.2 Example

3.2 Example

$$U_1 = 0 \qquad U_{n+1} = 3U_n + 1$$

U_1	U_2	U_3	U_4	U_5	U_6	U_7
0	1	4	13	40	121	364

[4 marks]

3.4.3 Solutions to 3.3 Exercise

Answer 1

$$A_1 = 2 \qquad A_{n+1} = 2A_n - 1$$

A_1	A_2	A_3	A_4	A_5	A_6	A_7
2	3	5	9	17	33	65

[4 marks]

Answer 2

$$H_1 = 11 \qquad H_{n+1} = 2A_n - 10$$

H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8
11	12	14	18	26	42	74	138

[4 marks]

Answer 3

$$P_1 = 5 \qquad P_{n+1} = 2P_n - 3$$

(i)

P_1	P_2	P_3	P_4	P_5	P_6
5	7	11	19	35	67

[4 marks]

(ii) The professor is not correct because 35 is not prime.
The factors of 35 are {1, 5, 7, 35} and a prime has exactly two factors.
Give a reason for your answer.

[2 marks]

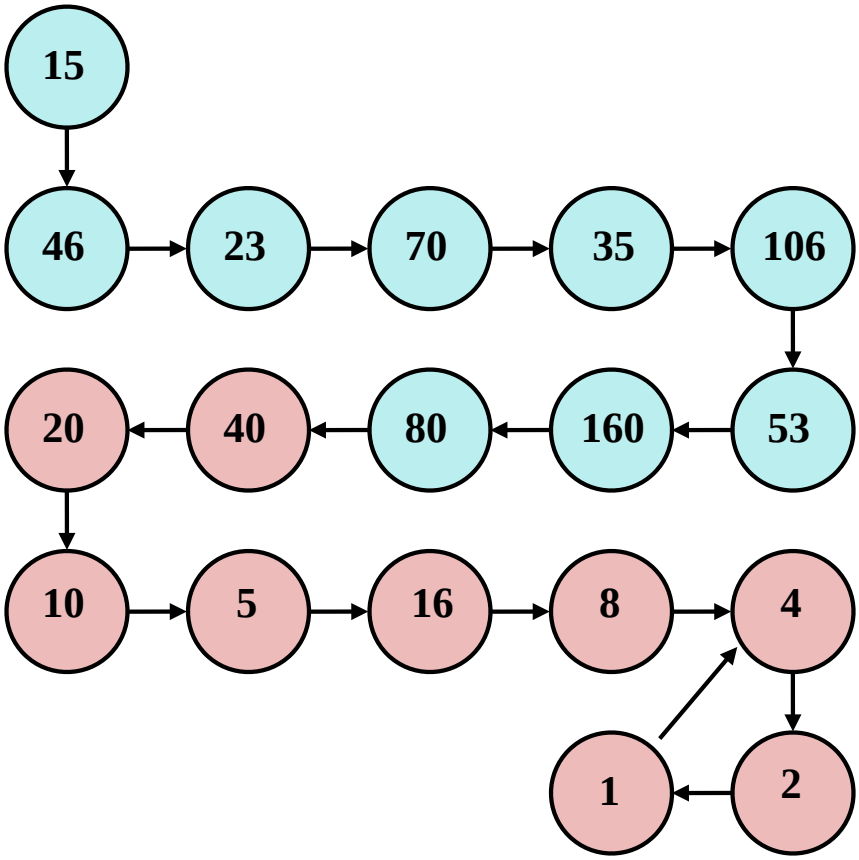
Answer 4

$$B_1 = 10 \qquad B_{n+1} = 3B_n - 10$$

B_1	B_2	B_3	B_4	B_5	B_6	B_7
10	20	50	140	410	1220	3650

[4 marks]

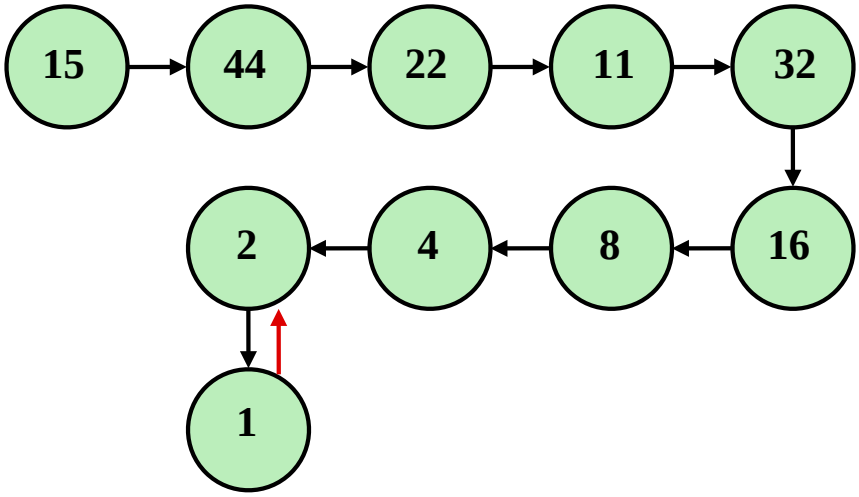
Answer 5



[6 marks]

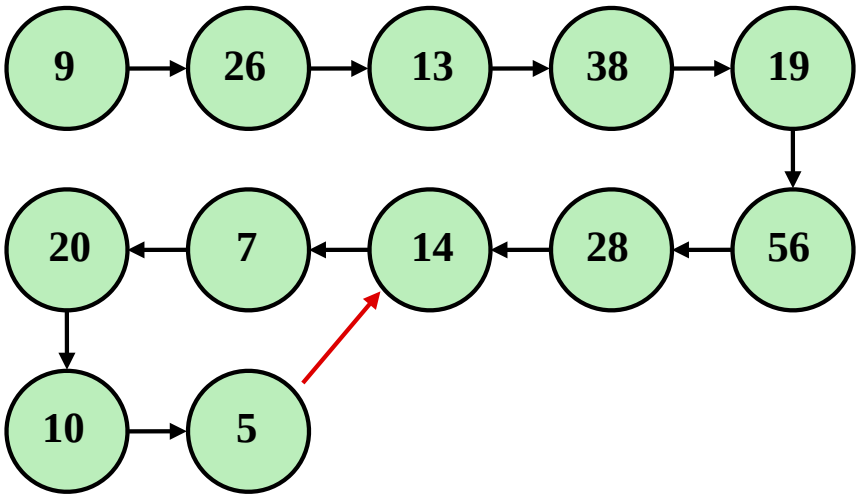
Answer 6

(i)



[6 marks]

(ii)



[6 marks]

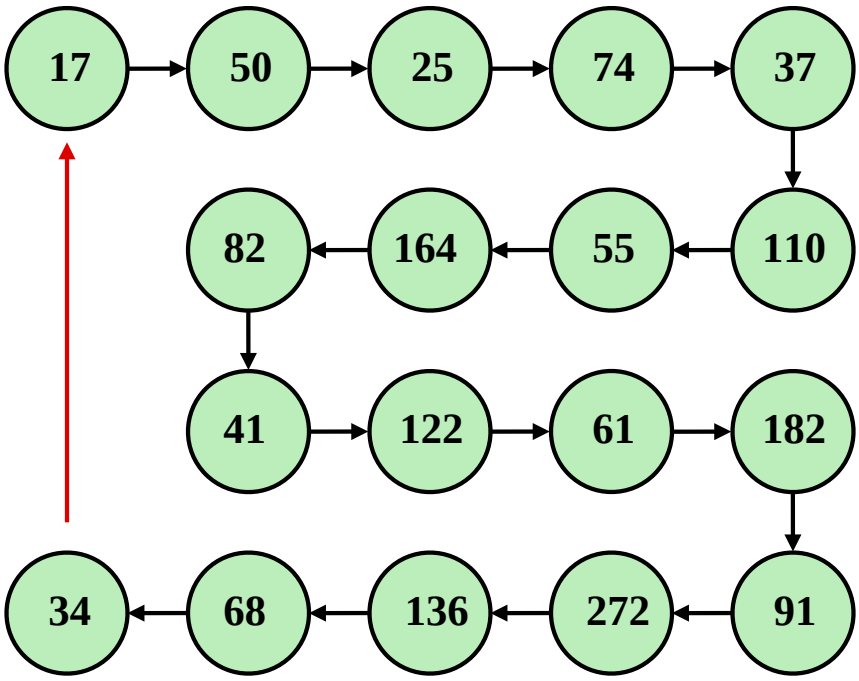
(iii) See the red arrow in the part (ii) answer, above.

[1 mark]

(iv) The part (ii) and (iii) answers show that for this adjusted function to Collatz conjecture is FALSE because the sequence has gone into an endless loop around the numbers 14, 7, 20, 10, and 5 and so does not ever get to the number 1

[2 marks]

(v)



[10 marks]

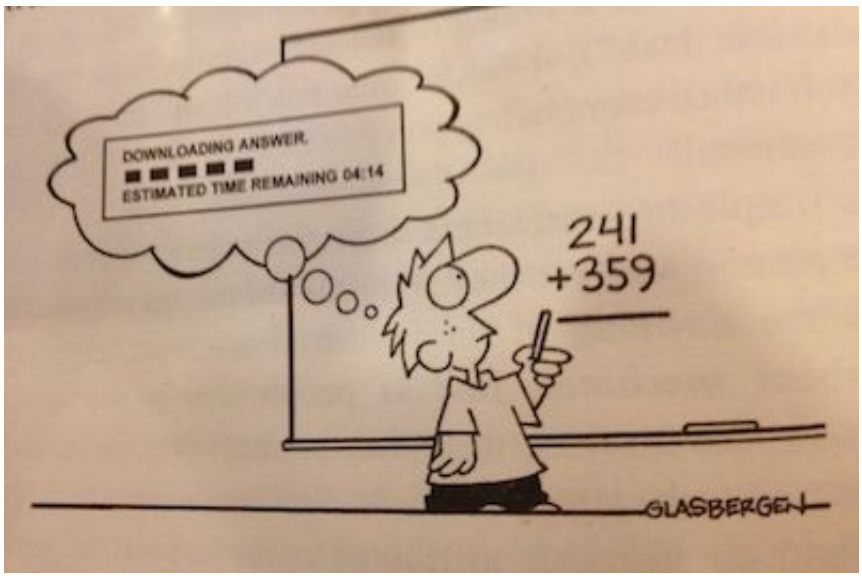
(vi) See the red arrow in the part (ii) answer, above.

[1 mark]

Lesson 4

GCSE Mathematics
Iteration

4.1 Fraction Frolics



To date, our iterations been restricted to numbers that are integers, $\mathbb{Z}$.

$$\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$$

However, in this lesson we will look at iterations that live amongst the rational numbers, $\mathbb{Q}$. Examples of rational numbers include,

$$\frac{2}{3}, \frac{7}{2}, \frac{6}{1}, -\frac{1}{3}, \frac{0}{1}$$

A rational number is a number that can be written as one integer divided by another (but not divided by zero, as division by zero is not allowed).

4.2 Examples #1
(Non-Calculator)

Calculate and simplify,

(i) $\frac{3}{7} \times \frac{4}{5}$ **(ii)** $\frac{5}{12} \times \frac{4}{7}$

(iii) $\frac{4}{7} \times 2$ **(iv)** $\frac{12}{13} \times 13$

[4 marks]

4.3 Examples #2
(Non-Calculator)

Calculate and simplify,

(i) $12\left(\frac{1}{4} + 2\right)$ **(ii)** $\left(\frac{4}{5} + \frac{1}{3}\right) \times 15$

(iii) $8\left(2 + \frac{3}{8}\right)$ **(iv)** $\left(\frac{3}{5} + 2\right) \times 5$

[4 marks]

4.4 An Iteration Involving Fractions

A sequence of numbers has the iterative rule

$$U_1 = \frac{3}{5} \qquad U_{n+1} = \frac{1}{2} U_n$$

Use the space below to work out the first six terms of this iterative sequence and put your answers in the table at the bottom of the page.

U_1	U_2	U_3	U_4	U_5	U_6

[4 marks]

4.5 Exercise

Non-Calculator

Marks Available : 50

Question 1

Calculate and simplify,

(i) $\frac{3}{11} \times \frac{5}{7}$ **(ii)** $\frac{4}{9} \times \frac{3}{5}$

(iii) $\frac{5}{13} \times 2$ **(iv)** $\frac{10}{17} \times 17$

[4 marks]

Question 2

First expand the brackets, then simplify,

(i) $15\left(\frac{1}{5} + 2\right)$ **(ii)** $\left(\frac{3}{4} + \frac{1}{6}\right) \times 12$

(iii) $7\left(3 + \frac{4}{7}\right)$ **(iv)** $\left(\frac{5}{6} + 2\right) \times 6$

[4 marks]

Question 3

First expand the brackets, then simplify,

(i) $\frac{2}{3}\left(2 + \frac{1}{2}\right)$ **(ii)** $\frac{3}{5}\left(\frac{4}{3} + 3\right)$

[4 marks]

Question 4

A sequence of numbers has the iterative rule

$$A_1 = \frac{1}{2} \qquad A_{n+1} = \frac{1}{2} A_n$$

Use the space below to work out the first six terms of this iterative sequence then put your answers in the table.

A_1	A_2	A_3	A_4	A_5	A_6

[6 marks]

Question 5

A sequence of numbers has the iterative rule

$$B_1 = \frac{16}{81} \qquad B_{n+1} = \frac{3}{2} B_n$$

Use the space below to work out the first eight terms of this iterative sequence then put your answers in the table.

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8

[6 marks]

Question 6

First expand the brackets, then simplify,

(i) $3\left(2 + \frac{1}{3}\right)$

[2 marks]

(ii) $\frac{3}{3} \times \frac{\left(2 + \frac{1}{3}\right)}{1}$

[2 marks]

Question 7

First expand the brackets, then simplify,

(i) $5\left(3 + \frac{2}{5}\right)$

[2 marks]

(ii) $\frac{5}{5} \times \frac{\left(3 + \frac{2}{5}\right)}{1}$

[2 marks]

Question 8

First expand the brackets, then simplify,

(i) $\left(2 + \frac{3}{7}\right) \times 7$

[2 marks]

(ii) $\frac{\left(2 + \frac{3}{7}\right)}{1} \times \frac{7}{7}$

[2 marks]

Question 9

Calculate $4 + \frac{3}{5}$ by

- Putting brackets around the question
- Putting the bracketed question all over 1
- Multiplying by $\frac{5}{5}$

[4 marks]

Question 10

Calculate $3 + \frac{4}{7}$ by

- Putting brackets around the question
- Putting the bracketed question all over 1
- Multiplying by $\frac{7}{7}$

[4 marks]

Question 11

A sequence of numbers has the iterative rule

$$G_1 = \frac{1}{4} \qquad G_{n+1} = \frac{2}{3} G_n$$

Use the space below to work out the first six terms of this iterative sequence then put your answers in the table.

G_1	G_2	G_3	G_4	G_5	G_6

[6 marks]

4.6 Answers

4.6.1 Solutions to 4.2 Examples #1

(i) $\frac{3}{7} \times \frac{4}{5} = \frac{12}{35}$ **(ii)** $\frac{5}{12} \times \frac{4}{7} = \frac{5}{3} \times \frac{1}{7} = \frac{5}{21}$

(iii) $\frac{4}{7} \times 2 = \frac{8}{7}$ **(iv)** $\frac{12}{13} \times 13 = 12$

[4 marks]

4.6.2 Solutions to 4.3 Examples #2

(i) $12\left(\frac{1}{4} + 2\right)$

$= 3 + 24$

$= 27$

(ii) $\left(\frac{4}{5} + \frac{1}{3}\right) \times 15$

$= 12 + 5$

$= 17$

(iii) $8\left(2 + \frac{3}{8}\right)$

$= 16 + 3$

$= 19$

(iv) $\left(\frac{3}{5} + 2\right) \times 5$

$= 3 + 10$

$= 13$

[4 marks]

4.6.3 Solution to 4.4 An Iteration Involving Fractions

$$U_1 = \frac{3}{5} \qquad U_{n+1} = \frac{1}{2} U_n$$

U_1	U_2	U_3	U_4	U_5	U_6
$\frac{3}{5}$	$\frac{3}{10}$	$\frac{3}{20}$	$\frac{3}{40}$	$\frac{3}{80}$	$\frac{3}{160}$

[4 marks]

4.6.4 Solutions to 4.5 Exercise

Answer 1

(i) $\frac{3}{11} \times \frac{5}{7} = \frac{15}{77}$ **(ii)** $\frac{4}{9} \times \frac{3}{5} = \frac{4}{3} \times \frac{1}{5} = \frac{4}{15}$

(iii) $\frac{5}{13} \times 2 = \frac{10}{13}$ **(iv)** $\frac{10}{17} \times 17 = 10$

[4 marks]

Answer 2

(i) $15\left(\frac{1}{5} + 2\right)$

$= 3 + 30$

$= 33$

(ii) $\left(\frac{3}{4} + \frac{1}{6}\right) \times 12$

$= 9 + 2$

$= 11$

(iii) $7\left(3 + \frac{4}{7}\right)$

$= 21 + 4$

$= 25$

(iv) $\left(\frac{5}{6} + 2\right) \times 6$

$= 5 + 12$

$= 17$

[4 marks]

Answer 3

(i) $\frac{2}{3}\left(2 + \frac{1}{2}\right)$

$= \frac{4}{3} + \frac{1}{3}$

$= \frac{5}{3}$

(ii) $\frac{3}{5}\left(\frac{4}{3} + 3\right)$

$= \frac{4}{5} + \frac{9}{5}$

$= \frac{13}{5}$

[4 marks]

Answer 4

$$A_1 = \frac{1}{2} \qquad A_{n+1} = \frac{1}{2} A_n$$

A_1	A_2	A_3	A_4	A_5	A_6
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$

[6 marks]

Answer 5

$$B_1 = \frac{16}{81} \qquad B_{n+1} = \frac{3}{2} B_n$$

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8
$\frac{16}{81}$	$\frac{8}{27}$	$\frac{4}{9}$	$\frac{2}{3}$	$\frac{1}{1}$	$\frac{3}{2}$	$\frac{9}{4}$	$\frac{27}{8}$

[6 marks]

Answer 6

(i) $3\left(2 + \frac{1}{3}\right) = 6 + 1 = 7$

[2 marks]

(ii) $\frac{3}{3} \times \frac{\left(2 + \frac{1}{3}\right)}{1} = \frac{6 + 1}{3} = \frac{7}{3}$

[2 marks]

Answer 7

(i) $5\left(3 + \frac{2}{5}\right) = 15 + 2 = 17$

[2 marks]

(ii) $\frac{5}{5} \times \frac{\left(3 + \frac{2}{5}\right)}{1} = \frac{15 + 2}{5} = \frac{17}{5}$

[2 marks]

Answer 8

(i) $\left(2 + \frac{3}{7}\right) \times 7 = 14 + 3 = 17$

[2 marks]

(ii) $\frac{\left(2 + \frac{3}{7}\right)}{1} \times \frac{7}{7} = \frac{14 + 3}{7} = \frac{17}{7}$

[2 marks]

Answer 9

$$\frac{\left(4 + \frac{3}{5}\right)}{1} \times \frac{5}{5} = \frac{20 + 3}{5} = \frac{23}{5}$$

[4 marks]

Answer 10

$$\frac{\left(3 + \frac{4}{7}\right)}{1} \times \frac{7}{7} = \frac{21 + 4}{7} = \frac{25}{7}$$

[4 marks]

Answer 11

$$G_1 = \frac{1}{4} \qquad G_{n+1} = \frac{2}{3} G_n$$

G_1	G_2	G_3	G_4	G_5	G_6
$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{2}{27}$	$\frac{4}{81}$	$\frac{8}{243}$

[6 marks]

Lesson 5

GCSE Mathematics
Iteration

5.1 The Many Faces of One

In mathematics, all numbers are important but some are more special than others. In today's lesson extensive use will be made of the fact that when anything is multiplied by one it does nothing.

You may be wondering why, if multiplying by one “does nothing”, it's of any use at all. The usefulness of one comes from the fact that it has many faces.

5.2 Example

Determine the first six terms of the sequence with term-to-term description,

$$U_1 = \frac{1}{3}, \quad U_{n+1} = \frac{2 - U_n}{2}$$

U_1	U_2	U_3	U_4	U_5	U_6

[7 marks]

5.3 Exercise

Non-Calculator

Marks Available : 50

Question 1

Simplify,

(i) $14 \times \frac{5}{7}$ **(ii)** $5\left(3 + \frac{1}{5}\right)$

(iii) $\left(\frac{5}{7} + 3\right) \times 7$ **(iv)** $\left(7 + \frac{3}{4}\right) \times 4$

[4 marks]

Question 2

Each of the following calculations has an answer that is a rational number.

That is, a number in the form $\frac{p}{q}$ for integer p and q with $q \neq 0$

For each, determine what that rational number is.

(i) $\frac{\left(1 + \frac{1}{2}\right)}{13} \times \frac{2}{2}$

[2 marks]

(ii) $\frac{\left(3 + \frac{1}{4}\right)}{7} \times \frac{4}{4}$

[2 marks]

(iii) $\frac{\left(\frac{5}{7} + 2\right)}{6} \times \frac{7}{7}$

[2 marks]

(iv) $\frac{\left(11 + \frac{5}{9}\right)}{7}$

[2 marks]

(v) $\frac{\left(4 + \frac{3}{8}\right)}{8}$

[2 marks]

(vi) $\frac{\left(\frac{8}{3} + 7\right)}{4}$

[2 marks]

Question 3

(i) The following sum has an answer that is a rational number.

That is, a number in the form $\frac{p}{q}$ for integer p and q with $q \neq 0$

Determine what that rational number is.

$$\frac{\left(3 - \frac{1}{2}\right)}{2}$$

[2 marks]

(ii) Consider the iteration, $A_1 = \frac{1}{2}$, $A_{n+1} = \frac{3 - A_n}{2}$

Use the space below to work out the first six terms of this iterative sequence and put your answers in the table towards the bottom of the page.

A_1	A_2	A_3	A_4	A_5	A_6

[7 marks]

(iii) From looking at your table of results, and spotting a pattern, write down what the next three terms, A_7, A_8 and A_9 are likely to be.

[2 marks]

Question 4

(i) The following sum has an answer that is a rational number.

That is, a number in the form $\frac{p}{q}$ for integer p and q with $q \neq 0$

Determine what that rational number is.

$$\frac{\left(1 - \frac{3}{10}\right)}{2}$$

[2 marks]

(ii) Consider the iteration, $B_1 = \frac{3}{10}$, $B_{n+1} = \frac{1 - B_n}{2}$

Use the space below to work out the first six terms of this iterative sequence and put your answers in the table towards the bottom of the page.

B_1	B_2	B_3	B_4	B_5	B_6

[7 marks]

(iii) From looking at your table of results, and spotting a pattern, write down what the next three terms, B_7, B_8 and B_9 are likely to be.

[2 marks]

Question 5

(i) The following sum has an answer that is a rational number.

That is, a number in the form $\frac{p}{q}$ for integer p and q with $q \neq 0$

Determine what that rational number is.

$$\frac{\left(1 - \frac{2}{3}\right)}{3}$$

[2 marks]

(ii) Consider the iteration, $Z_1 = \frac{2}{3}$, $Z_{n+1} = \frac{1 - Z_n}{3}$

Use the space below to work out the first six terms of this iterative sequence and put your answers in the table towards the bottom of the page.

Z_1	Z_2	Z_3	Z_4	Z_5	Z_6

[7 marks]

(iii) From looking at your table of results, and spotting a pattern, write down what the next three terms, Z_7, Z_8 and Z_9 are likely to be.

[3 marks]

5.4 Answers

5.4.1 Solutions to 5.2 Example

$$U_1 = \frac{1}{3}, \quad U_{n+1} = \frac{2 - U_n}{2}$$

$$U_2 = \frac{\left(2 - \frac{1}{3}\right)}{2} \times \frac{3}{3} = \frac{6 - 1}{6} = \frac{5}{6}$$

$$U_3 = \frac{\left(2 - \frac{5}{6}\right)}{2} \times \frac{6}{6} = \frac{12 - 5}{12} = \frac{7}{12}$$

$$U_4 = \frac{\left(2 - \frac{7}{12}\right)}{2} \times \frac{12}{12} = \frac{24 - 7}{24} = \frac{17}{24}$$

$$U_5 = \frac{\left(2 - \frac{17}{24}\right)}{2} \times \frac{24}{24} = \frac{48 - 17}{48} = \frac{31}{48}$$

$$U_6 = \frac{\left(2 - \frac{31}{48}\right)}{2} \times \frac{48}{48} = \frac{96 - 31}{96} = \frac{65}{96}$$

U_1	U_2	U_3	U_4	U_5	U_6
$\frac{1}{3}$	$\frac{5}{6}$	$\frac{7}{12}$	$\frac{17}{24}$	$\frac{31}{48}$	$\frac{65}{96}$

[6 marks]

5.4.2 Solutions to 5.3 Exercise

Answer 1

(i) $14 \times \frac{5}{7} = 10$

$= 2 \times 5$

$= 10$

(ii) $5\left(3 + \frac{1}{5}\right)$

$= 15 + 1$

$= 16$

(iii) $\left(\frac{5}{7} + 3\right) \times 7$

$= 5 + 21$

$= 26$

(iv) $\left(7 + \frac{3}{4}\right) \times 4$

$= 28 + 3$

$= 31$

[4 marks]

Answer 2

(i) $\dfrac{\left(1 + \frac{1}{2}\right)}{13} \times \dfrac{2}{2} = \dfrac{2 + 1}{26} = \dfrac{3}{26}$

[2 marks]

(ii) $\dfrac{\left(3 + \frac{1}{4}\right)}{7} \times \dfrac{4}{4} = \dfrac{12 + 1}{28} = \dfrac{13}{28}$

[2 marks]

(iii) $\dfrac{\left(\frac{5}{7} + 2\right)}{6} \times \dfrac{7}{7} = \dfrac{5 + 14}{42} = \dfrac{19}{42}$

[2 marks]

(iv) $\dfrac{\left(11 + \frac{5}{9}\right)}{7} \times \dfrac{9}{9} = \dfrac{99 + 5}{63} = \dfrac{104}{63}$

[2 marks]

(v) $\dfrac{\left(4 + \frac{3}{8}\right)}{8} \times \dfrac{8}{8} = \dfrac{32 + 3}{64} = \dfrac{35}{64}$

[2 marks]

(vi) $\dfrac{\left(\frac{8}{3} + 7\right)}{4} \times \dfrac{3}{3} = \dfrac{8 + 21}{12} = \dfrac{29}{12}$

[2 marks]

Answer 3

(i) $\dfrac{\left(3 - \frac{1}{2}\right)}{2} \times \dfrac{2}{2} = \dfrac{6 - 1}{4} = \dfrac{5}{4}$

(ii) $A_1 = \dfrac{1}{2}$, $A_{n+1} = \dfrac{3 - A_n}{2}$

$A_2 = \dfrac{5}{4}$ (from part (i))

$A_3 = \dfrac{\left(3 - \frac{5}{4}\right)}{2} \times \dfrac{4}{4} = \dfrac{12 - 5}{8} = \dfrac{7}{8}$

$A_4 = \dfrac{\left(3 - \frac{7}{8}\right)}{2} \times \dfrac{8}{8} = \dfrac{24 - 7}{16} = \dfrac{17}{16}$

$A_5 = \dfrac{\left(3 - \frac{17}{16}\right)}{2} \times \dfrac{16}{16} = \dfrac{48 - 17}{32} = \dfrac{31}{32}$

$A_6 = \dfrac{\left(3 - \frac{31}{32}\right)}{2} \times \dfrac{32}{32} = \dfrac{96 - 31}{64} = \dfrac{65}{64}$

A_1	A_2	A_3	A_4	A_5	A_6
$\frac{1}{2}$	$\frac{5}{4}$	$\frac{7}{8}$	$\frac{17}{16}$	$\frac{31}{32}$	$\frac{65}{64}$

[7 marks]

(iii) The denominators are powers of 2

The numerators are alternatively one more or one less than the denominator

$A_7 = \dfrac{127}{128}$ $A_8 = \dfrac{257}{256}$ $A_9 = \dfrac{511}{512}$

[2 marks]

Answer 4

(i) $\dfrac{\left(1 - \frac{3}{10}\right)}{2} \times \dfrac{10}{10} = \dfrac{10 - 3}{20} = \dfrac{7}{20}$

(ii) $B_1 = \dfrac{3}{10}$, $B_{n+1} = \dfrac{1 - B_n}{2}$

$B_2 = \dfrac{7}{20}$ (from part (i))

$B_3 = \dfrac{\left(1 - \frac{7}{20}\right)}{2} \times \dfrac{20}{20} = \dfrac{20 - 7}{40} = \dfrac{13}{40}$

$B_4 = \dfrac{\left(1 - \frac{13}{40}\right)}{2} \times \dfrac{40}{40} = \dfrac{40 - 13}{80} = \dfrac{27}{80}$

$B_5 = \dfrac{\left(1 - \frac{27}{80}\right)}{2} \times \dfrac{80}{80} = \dfrac{80 - 27}{160} = \dfrac{53}{160}$

$B_6 = \dfrac{\left(1 - \frac{53}{160}\right)}{2} \times \dfrac{160}{160} = \dfrac{160 - 53}{320} = \dfrac{107}{320}$

B_1	B_2	B_3	B_4	B_5	B_6
$\frac{3}{10}$	$\frac{7}{20}$	$\frac{13}{40}$	$\frac{27}{80}$	$\frac{53}{160}$	$\frac{107}{320}$

[7 marks]

(iii) Each denominators is double the previous denominator

Each numerator is previous denominator minus previous numerator

$B_7 = \dfrac{213}{640}$ $B_8 = \dfrac{427}{1280}$ $B_9 = \dfrac{853}{1560}$

[2 marks]

Answer 5

(i) $\dfrac{\left(1-\frac{2}{3}\right)}{3} \times \dfrac{3}{3} = \dfrac{3-2}{9} = \dfrac{1}{9}$

(ii) $Z_1 = \dfrac{2}{3}$, $Z_{n+1} = \dfrac{1-Z_n}{3}$

$Z_2 = \dfrac{1}{9}$ (from part (i))

$Z_3 = \dfrac{\left(1-\frac{1}{9}\right)}{3} \times \dfrac{9}{9} = \dfrac{9-1}{27} = \dfrac{8}{27}$

$Z_4 = \dfrac{\left(1-\frac{8}{27}\right)}{3} \times \dfrac{27}{27} = \dfrac{27-8}{81} = \dfrac{19}{81}$

$Z_5 = \dfrac{\left(1-\frac{19}{81}\right)}{3} \times \dfrac{81}{81} = \dfrac{81-19}{243} = \dfrac{62}{243}$

$Z_6 = \dfrac{\left(1-\frac{62}{243}\right)}{3} \times \dfrac{243}{243} = \dfrac{243-62}{729} = \dfrac{181}{729}$

Z_1	Z_2	Z_3	Z_4	Z_5	Z_6
$\frac{2}{3}$	$\frac{1}{9}$	$\frac{8}{27}$	$\frac{19}{81}$	$\frac{62}{243}$	$\frac{181}{729}$

[7 marks]

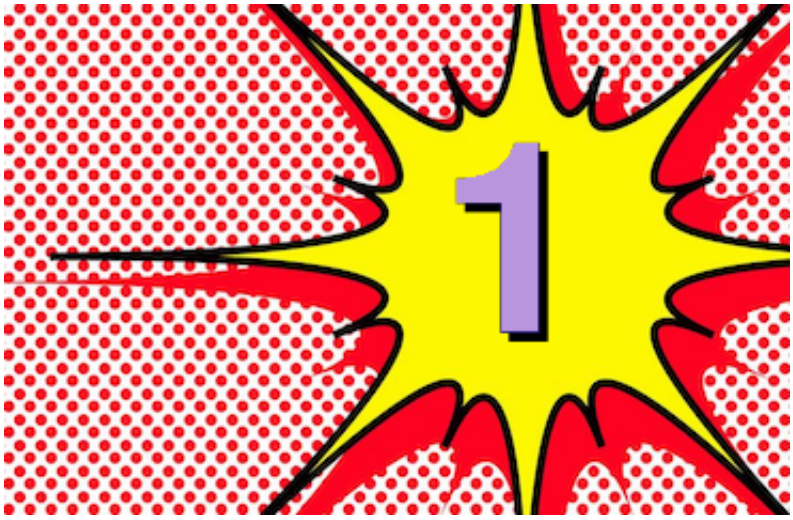
(iii) Each denominators is triple the previous denominator

Each numerator is previous denominator minus previous numerator

$Z_7 = \dfrac{548}{2187}$ $Z_8 = \dfrac{1639}{6561}$ $Z_9 = \dfrac{4922}{19683}$

[2 marks]

6.1 Around and Around



Last lesson we started to have a deeper appreciation of just how useful the number one can be in allowing us to handle some tricky calculations involving fractions without resorting to the use of a calculator. In this lesson we look at a variation of the same theme. Once again, the objective is to go around and around in an iterative loop.

6.2 Example

Without using a calculator, simplify the following,

$$\frac{1}{\left(\frac{3}{4}+1\right)}$$

There are many ways of tackling this but we are going to seemingly randomly, multiply it by one. This is allowed because it does not change anything. But although it's allowed why do it ?

$$\frac{1}{\left(\frac{3}{4}+1\right)} \times 1$$

The 4 is coloured red because we're going to use the red number method of doing the calculation. This involves picking the red face of the number one.

[2 marks]

6.3 Example

This work is still non-calculator, so we will make heavy use of the red number method and the many faces of the number one to not get into a muddle with all of the fractions involved.

Consider the iteration, $F_1 = 3$, $F_{n+1} = \dfrac{2}{F_n + 1}$

Use the space below to work out the first five terms of this iterative sequence and put your answers in the table at the bottom of the page.

F_1	F_2	F_3	F_4	F_5	F_6

[7 marks]

6.4 Exercise

Non-Calculator

Marks Available : 40

Question 1

Each of the following calculations has an answer that is a rational number.

That is, a number in the form $\frac{p}{q}$ for integer p and q with $q \neq 0$

For each, determine what that rational number is.

(i) $\dfrac{1}{\left(\frac{2}{3} + 5\right)} \times \dfrac{3}{3}$

[2 marks]

(ii) $\dfrac{4}{\left(\frac{5}{2} + 3\right)} \times \dfrac{2}{2}$

[2 marks]

(iii) $\dfrac{3}{\left(\frac{9}{4} - 1\right)} \times \dfrac{4}{4}$

[2 marks]

(iv) $\dfrac{4}{\left(\frac{8}{13} + 3\right)} \times \dfrac{13}{13}$

[2 marks]

Question 2

Each of the following calculations has an answer that is a rational number.

That is, a number in the form $\frac{p}{q}$ for integer p and q with $q \neq 0$

For each, determine what that rational number is.

Use the "red number" technique suggested by question 1.

(i) $\dfrac{1}{\left(\frac{4}{3} + 2\right)}$

[2 marks]

(ii) $\dfrac{5}{\left(\frac{3}{7} + 2\right)}$

[2 marks]

(iii) $\dfrac{4}{\left(\frac{23}{20} + 5\right)}$

[2 marks]

(iv) $\dfrac{7}{\left(\frac{100}{101} + 1\right)}$

[2 marks]

Question 3

(i) The following sum has an answer that is a rational number.

That is, a number in the form $\frac{p}{q}$ for integer p and q with $q \neq 0$

Determine what that rational number is.

$$\frac{1}{\left(\frac{1}{2} + 1\right)}$$

[2 marks]

(ii) Consider the iteration, $A_1 = \frac{1}{2}$, $A_{n+1} = \frac{1}{A_n + 1}$

Use the space below to work out the first six terms of this iterative sequence and put your answers in the table towards the bottom of the page.

A_1	A_2	A_3	A_4	A_5	A_6

[7 marks]

(iii) From looking at your table of results, and spotting a pattern, write down what the next three terms, A_7, A_8 and A_9 are likely to be.

[2 marks]

Question 4

(i) The following sum has an answer that is a rational number.

That is, a number in the form $\frac{p}{q}$ for integer p and q with $q \neq 0$

Determine what that rational number is.

$$\frac{1}{\left(\frac{2}{3} + 1\right)}$$

[2 marks]

(ii) Consider the iteration, $B_1 = \frac{1}{3}$, $B_{n+1} = \frac{1}{2B_n + 1}$

Use the space below to work out the first six terms of this iterative sequence and put your answers in the table towards the bottom of the page.

B_1	B_2	B_3	B_4	B_5	B_6

[7 marks]

(iii) From looking at your table of results, and spotting a pattern, write down what the next three terms, B_7, B_8 and B_9 are likely to be.

[2 marks]

Question 5

(i) The following sum has an answer that is a rational number.

That is, a number in the form $\frac{p}{q}$ for integer p and q with $q \neq 0$

Determine what that rational number is.

$$\frac{1}{\left(\frac{3}{5} + 2\right)}$$

[2 marks]

(ii) Consider the iteration, $Z_1 = \frac{1}{5}$, $Z_{n+1} = \frac{1}{3Z_n + 2}$

Use the space below to work out the first six terms of this iterative sequence and put your answers in the table towards the bottom of the page.

Z_1	Z_2	Z_3	Z_4	Z_5	Z_6

[8 marks]

(iii) From looking at your table of results, and spotting a pattern, write down what the next three terms, Z_7, Z_8 and Z_9 are likely to be.

[2 marks]

6.5 Answers

GCSE Mathematics
Iteration

6.5.1 Solution to 5.2 Example

$$\frac{1}{\left(\frac{3}{4}+1\right)} \times \frac{4}{4} = \frac{4}{3+4} = \frac{4}{7}$$

[2 marks]

6.5.2 Solution to 5.3 Example

When $n = 1 \Rightarrow F_2 = \dfrac{2}{F_1 + 1} = \dfrac{2}{3+1}$

$$= \frac{1}{2}$$

When $n = 2 \Rightarrow F_3 = \dfrac{2}{F_2 + 1} = \dfrac{2}{\left(\frac{1}{2}+1\right)} \times \dfrac{2}{2}$

$$= \frac{4}{1+2}$$

$$= \frac{4}{3}$$

When $n = 3 \Rightarrow F_4 = \dfrac{2}{F_3 + 1} = \dfrac{2}{\left(\frac{4}{3}+1\right)} \times \dfrac{3}{3}$

$$= \frac{6}{4+3}$$

$$= \frac{6}{7}$$

When $n = 4 \Rightarrow F_5 = \dfrac{2}{F_4 + 1} = \dfrac{2}{\left(\frac{6}{7}+1\right)} \times \dfrac{7}{7}$

$$= \frac{14}{6+7}$$

$$= \frac{14}{13}$$

When $n = 5 \Rightarrow F_6 = \dfrac{2}{F_5 + 1} = \dfrac{2}{\left(\frac{14}{13}+1\right)} \times \dfrac{13}{13}$

$$= \frac{26}{14+13}$$

$$= \frac{26}{27}$$

F_1	F_2	F_3	F_4	F_5	F_6
$\frac{1}{1}$	$\frac{1}{2}$	$\frac{4}{3}$	$\frac{6}{7}$	$\frac{14}{13}$	$\frac{26}{27}$

[7 marks]

6.5.3 Solutions to Exercise 6.4

Answer 1

(i) $\dfrac{1}{\left(\frac{2}{3}+5\right)} \times \dfrac{3}{3} = \dfrac{3}{2+15} = \dfrac{3}{17}$

[2 marks]

(ii) $\dfrac{4}{\left(\frac{5}{2}+3\right)} \times \dfrac{2}{2} = \dfrac{8}{5+6} = \dfrac{8}{11}$

[2 marks]

(iii) $\dfrac{3}{\left(\frac{9}{4}-1\right)} \times \dfrac{4}{4} = \dfrac{12}{9-4} = \dfrac{12}{5}$

[2 marks]

(iv) $\dfrac{4}{\left(\frac{8}{13}+3\right)} \times \dfrac{13}{13} = \dfrac{52}{8+39} = \dfrac{52}{47}$

[2 marks]

Answer 2

(i) $\dfrac{1}{\left(\frac{4}{3}+2\right)} \times \dfrac{3}{3} = \dfrac{3}{4+6} = \dfrac{3}{10}$

[2 marks]

(ii) $\dfrac{5}{\left(\frac{3}{7}+2\right)} \times \dfrac{7}{7} = \dfrac{35}{3+14} = \dfrac{35}{17}$

[2 marks]

(iii) $\dfrac{4}{\left(\frac{23}{20}+5\right)} \times \dfrac{20}{20} = \dfrac{80}{23+100} = \dfrac{80}{123}$

[2 marks]

(iv) $\dfrac{7}{\left(\frac{100}{101}+1\right)} \times \dfrac{101}{101} = \dfrac{707}{100+101} = \dfrac{707}{201}$

[2 marks]

Answer 3

(i) $\dfrac{1}{\left(\frac{1}{2}+1\right)} \times \dfrac{2}{2} = \dfrac{2}{1+2} = \dfrac{2}{3}$

[2 marks]

(ii) $A_1 = \dfrac{1}{2}$, $A_{n+1} = \dfrac{1}{A_n + 1}$

$A_2 = \dfrac{2}{3}$ (from part (i))

$A_3 = \dfrac{1}{\left(\frac{2}{3}+1\right)} \times \dfrac{3}{3} = \dfrac{3}{2+3} = \dfrac{3}{5}$

$A_4 = \dfrac{1}{\left(\frac{3}{5}+1\right)} \times \dfrac{5}{5} = \dfrac{5}{3+5} = \dfrac{5}{8}$

$A_5 = \dfrac{1}{\left(\frac{5}{8}+1\right)} \times \dfrac{8}{8} = \dfrac{8}{5+8} = \dfrac{8}{13}$

$A_6 = \dfrac{1}{\left(\frac{8}{13}+1\right)} \times \dfrac{13}{13} = \dfrac{13}{8+13} = \dfrac{13}{21}$

A_1	A_2	A_3	A_4	A_5	A_6
$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{5}{8}$	$\frac{8}{13}$	$\frac{13}{21}$

[7 marks]

(iii) It's all to do with the Fibonacci numbers !

$A_7 = \dfrac{21}{34}$ $A_8 = \dfrac{34}{55}$ $A_9 = \dfrac{55}{89}$

[2 marks]

Answer 4

(i) $\dfrac{1}{\left(\frac{2}{3}+1\right)} \times \dfrac{3}{3} = \dfrac{3}{2+3} = \dfrac{3}{5}$

[2 marks]

(ii) $B_1 = \dfrac{1}{3}$, $B_{n+1} = \dfrac{1}{2B_n + 1}$

$B_2 = \dfrac{3}{5}$ (from part (i))

$B_3 = \dfrac{1}{\left(\frac{6}{5}+1\right)} \times \dfrac{5}{5} = \dfrac{5}{6+5} = \dfrac{5}{11}$

$B_4 = \dfrac{1}{\left(\frac{10}{11}+1\right)} \times \dfrac{11}{11} = \dfrac{11}{10+11} = \dfrac{11}{21}$

$B_5 = \dfrac{1}{\left(\frac{22}{21}+1\right)} \times \dfrac{21}{21} = \dfrac{21}{22+21} = \dfrac{21}{43}$

$B_6 = \dfrac{1}{\left(\frac{42}{43}+1\right)} \times \dfrac{43}{43} = \dfrac{43}{42+43} = \dfrac{43}{85}$

B_1	B_2	B_3	B_4	B_5	B_6
$\frac{1}{3}$	$\frac{3}{5}$	$\frac{5}{11}$	$\frac{11}{21}$	$\frac{21}{43}$	$\frac{43}{85}$

[7 marks]

(iii) Let $B_n = \dfrac{P_n}{Q_n}$ in which case $B_{n+1} = \begin{cases} \dfrac{Q_n}{2Q_n+1} & \text{if } n \text{ is even} \\ \dfrac{Q_n}{2Q_n-1} & \text{if } n \text{ is odd} \end{cases}$

$B_7 = \dfrac{85}{171}$ $B_8 = \dfrac{171}{341}$ $B_9 = \dfrac{341}{683}$

[2 marks]

Answer 5

(i) $\dfrac{1}{\left(\frac{3}{5}+2\right)} \times \dfrac{5}{5} = \dfrac{5}{3+10} = \dfrac{5}{13}$

[2 marks]

(ii) $Z_1 = \dfrac{1}{5}$, $Z_{n+1} = \dfrac{1}{3Z_n + 2}$

$Z_2 = \dfrac{5}{13}$ (from part (i))

$Z_3 = \dfrac{1}{\left(\frac{15}{13}+2\right)} \times \dfrac{13}{13} = \dfrac{13}{15+26} = \dfrac{13}{41}$

$Z_4 = \dfrac{1}{\left(\frac{39}{41}+2\right)} \times \dfrac{41}{41} = \dfrac{41}{39+82} = \dfrac{41}{121}$

$Z_5 = \dfrac{1}{\left(\frac{123}{121}+2\right)} \times \dfrac{121}{121} = \dfrac{121}{123+242} = \dfrac{121}{365}$

$Z_6 = \dfrac{1}{\left(\frac{363}{365}+2\right)} \times \dfrac{365}{365} = \dfrac{365}{363+730} = \dfrac{365}{1093}$

Z_1	Z_2	Z_3	Z_4	Z_5	Z_6
$\frac{1}{5}$	$\frac{5}{13}$	$\frac{13}{41}$	$\frac{41}{121}$	$\frac{121}{365}$	$\frac{365}{1093}$

[7 marks]

(iii) Let $Z_n = \dfrac{P_n}{Q_n}$ in which case $Z_{n+1} = \begin{cases} \dfrac{Q_n}{3Q_n+2} & \text{if } n \text{ is even} \\ \dfrac{Q_n}{3Q_n-2} & \text{if } n \text{ is odd} \end{cases}$

$Z_7 = \dfrac{1093}{3119}$ $Z_8 = \dfrac{3119}{9355}$ $Z_9 = \dfrac{9355}{28067}$

[2 marks]

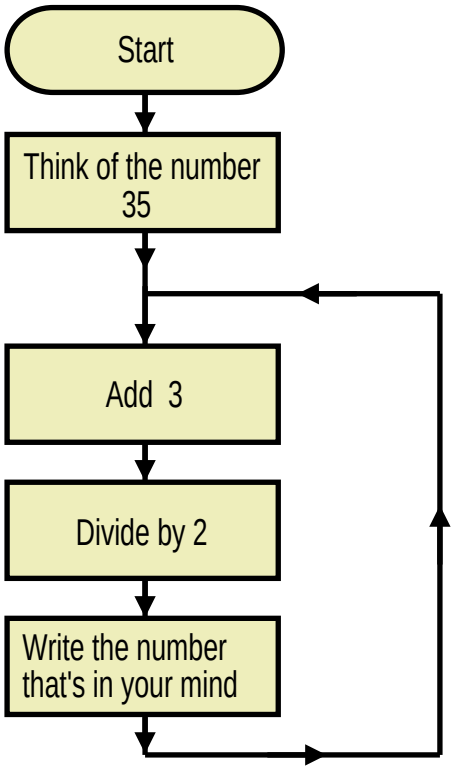
Lesson 7

GCSE Mathematics
Iteration

7.1 Revision

Non-Calculator
Marks Available : 60

Question 1
A number sequence, *U*, is described by the following flowchart,



Complete this table to show the first six terms in sequence *U*

U_1	U_2	U_3	U_4	U_5	U_6
35					

[7 marks]

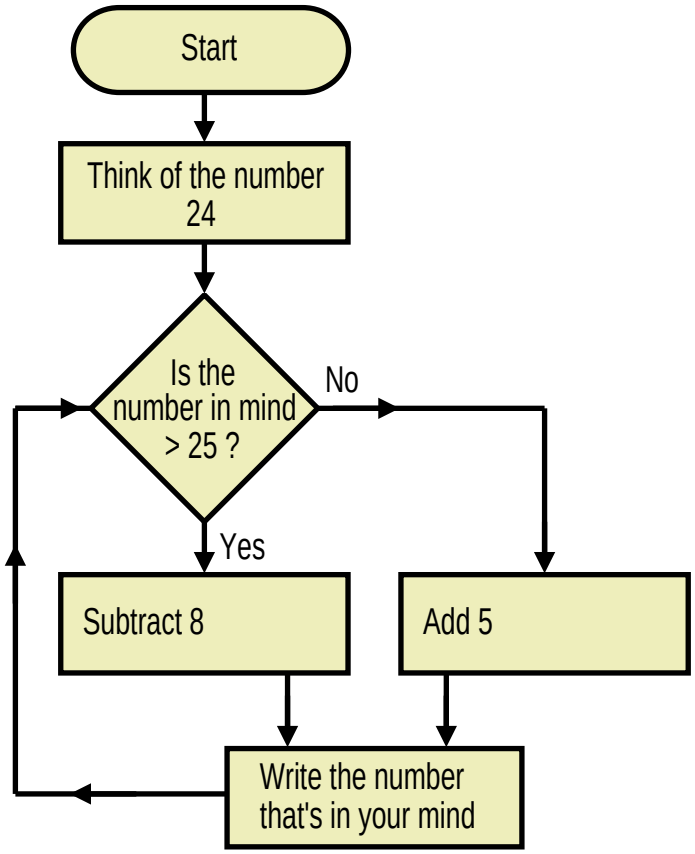
Question 2
Simplify,

(i) $9 \times \left(\frac{5}{9} + 3 \right)$ **(ii)** $\left(8 + \frac{3}{4} \right) \times 4$

[4 marks]

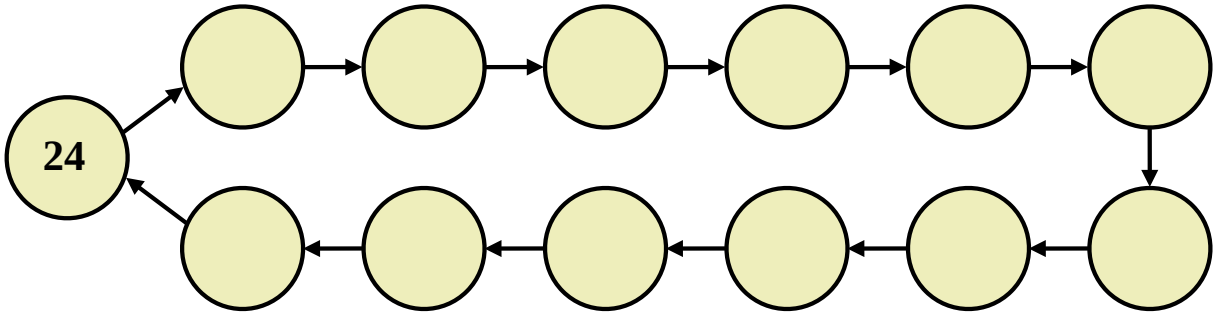
Question 3

A number sequence, *K*, is described by the following flowchart,



The flowchart generates a loop of numbers.

On the following diagram write out the numbers that are in the loop.



[8 marks]

Question 4

First expand the brackets, then simplify,

(i) $\left(3 + \frac{2}{7}\right) \times 7$

[2 marks]

(ii) $\frac{\left(3 + \frac{2}{7}\right)}{1} \times \frac{7}{7}$

[2 marks]

Question 5

A sequence of numbers has the iterative rule

$$A_1 = \frac{1}{9} \qquad A_{n+1} = \frac{3}{2} A_n$$

Use the space below to work out the first six terms of this iterative sequence.
Simplify fractions where possible.
Put your answers in the table.

A_1	A_2	A_3	A_4	A_5	A_6

[7 marks]

Question 6

Simplify,

(i) $\dfrac{\left(\frac{3}{11} + 2\right)}{6} \times \dfrac{11}{11}$

[2 marks]

(ii) $\dfrac{\left(5 + \frac{7}{9}\right)}{7}$

[2 marks]

(iii) $\dfrac{\left(4 + \frac{3}{7}\right)}{7}$

[2 marks]

Question 7

(i) The following sum has an answer that is a rational number.

That is, a number in the form $\frac{p}{q}$ for integer p and q with $q \neq 0$

Determine what that rational number is.

$$\frac{\left(4 - \frac{3}{2}\right)}{2}$$

[2 marks]

(ii) Consider the iteration, $B_1 = 1$, $B_{n+1} = \frac{4 - B_n}{2}$

Use the space below to work out the first six terms of this iterative sequence and put your answers in the table towards the bottom of the page.

B_1	B_2	B_3	B_4	B_5	B_6

[7 marks]

Question 8

Simplify,

(i) $\frac{1}{\left(\frac{5}{3} + 1\right)} \times \frac{3}{3}$

[2 marks]

(ii) $\frac{4}{\left(\frac{2}{7} + 3\right)}$

[2 marks]

(iii) $\frac{2}{\left(\frac{7}{10} + 5\right)}$

[2 marks]

Question 9

(i) The following sum has an answer that is a rational number.

That is, a number in the form $\frac{p}{q}$ for integer p and q with $q \neq 0$

Determine what that rational number is.

$$\frac{1}{\left(\frac{2}{5} + 1\right)}$$

[2 marks]

(ii) Consider the iteration, $Z_1 = 4$, $Z_{n+1} = \frac{2}{Z_n + 1}$

Use the space below to work out the first six terms of this iterative sequence and put your answers in the table towards the bottom of the page.

Z_1	Z_2	Z_3	Z_4	Z_5	Z_6

[7 marks]

7.2 Answers

GCSE Mathematics
Iteration

Answer 1

$$U_1 = 35, \quad U_{n+1} = \frac{U_n + 3}{2}$$

U_1	U_2	U_3	U_4	U_5	U_6
35	19	11	7	5	4

[7 marks]

Answer 2

(i) $9 \times \left(\frac{5}{9} + 3 \right)$

$= 5 + 27$

$= 32$

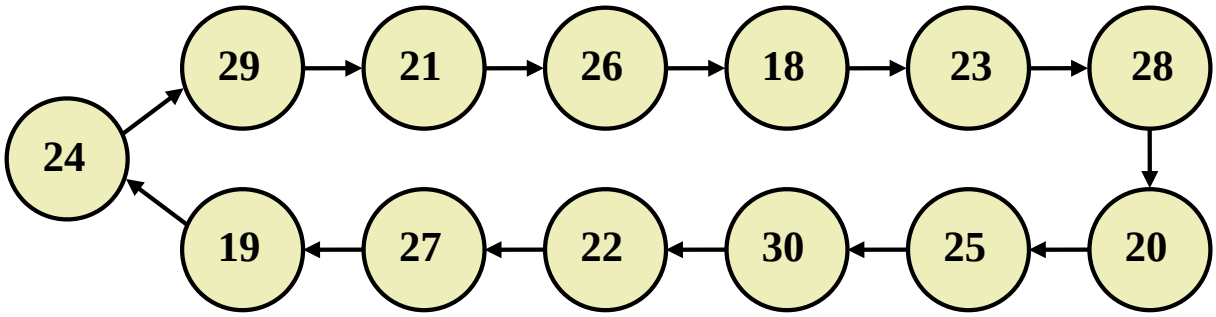
(ii) $\left(8 + \frac{3}{4} \right) \times 4$

$= 32 + 3$

$= 35$

[4 marks]

Answer 3



[8 marks]

Answer 4

(i) $\left(3 + \frac{2}{7} \right) \times 7 = 21 + 2 = 23$

[2 marks]

(ii) $\frac{\left(3 + \frac{2}{7} \right)}{1} \times \frac{7}{7} = \frac{21 + 2}{7} = \frac{23}{7}$

[2 marks]

Answer 5

$A_1 = \frac{1}{9} \qquad A_{n+1} = \frac{3}{2} A_n$

$A_2 = \frac{3}{2} \times \frac{1}{9} = \frac{1}{6}$

$A_3 = \frac{3}{2} \times \frac{1}{6} = \frac{1}{4}$

$A_4 = \frac{3}{2} \times \frac{1}{4} = \frac{3}{8}$

$A_5 = \frac{3}{2} \times \frac{3}{8} = \frac{9}{16}$

$A_6 = \frac{3}{2} \times \frac{9}{16} = \frac{27}{32}$

A_1	A_2	A_3	A_4	A_5	A_6
$\frac{1}{9}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{9}{16}$	$\frac{27}{32}$

[7 marks]

Answer 6

(i) $\frac{\left(\frac{3}{11} + 2\right)}{6} \times \frac{11}{11} = \frac{3 + 22}{66} = \frac{25}{66}$

[2 marks]

(ii) $\frac{\left(5 + \frac{7}{9}\right)}{7} \times \frac{9}{9} = \frac{45 + 7}{63} = \frac{52}{63}$

[2 marks]

(iii) $\frac{\left(4 + \frac{3}{7}\right)}{7} \times \frac{7}{7} = \frac{28 + 3}{49} = \frac{31}{49}$

[2 marks]

Answer 7

(i) $\dfrac{\left(4 - \frac{3}{2}\right)}{2} \times \dfrac{2}{2} = \dfrac{8 - 3}{4} = \dfrac{5}{4}$

[2 marks]

(ii) $B_1 = 1,\ B_{n+1} = \dfrac{4 - B_n}{2}$

$B_2 = \dfrac{4 - 1}{2} = \dfrac{3}{2}$

$B_3 = \dfrac{5}{4}$ (from part (i))

$B_4 = \dfrac{\left(4 - \frac{5}{4}\right)}{2} \times \dfrac{4}{4} = \dfrac{16 - 5}{8} = \dfrac{11}{8}$

$B_5 = \dfrac{\left(4 - \frac{11}{8}\right)}{2} \times \dfrac{8}{8} = \dfrac{32 - 11}{16} = \dfrac{21}{16}$

$B_6 = \dfrac{\left(4 - \frac{21}{16}\right)}{2} \times \dfrac{16}{16} = \dfrac{64 - 21}{32} = \dfrac{43}{32}$

B_1	B_2	B_3	B_4	B_5	B_6
1	$\frac{3}{2}$	$\frac{5}{4}$	$\frac{11}{8}$	$\frac{21}{16}$	$\frac{43}{32}$

[7 marks]

Answer 8

(i) $\dfrac{1}{\left(\frac{5}{3} + 1\right)} \times \dfrac{3}{3} = \dfrac{3}{5 + 3} = \dfrac{3}{8}$

[2 marks]

(ii) $\dfrac{4}{\left(\frac{2}{7} + 3\right)} \times \dfrac{7}{7} = \dfrac{28}{2 + 21} = \dfrac{28}{23}$

[2 marks]

(iii) $\dfrac{2}{\left(\frac{7}{10} + 5\right)} \times \dfrac{10}{10} = \dfrac{20}{7 + 50} = \dfrac{20}{57}$

[2 marks]

Answer 9

(i) $$\frac{2}{\left(\frac{2}{5}+1\right)} \times \frac{5}{5} = \frac{10}{2+5} = \frac{10}{7}$$

[2 marks]

(ii) $$Z_1 = 4,\ Z_{n+1} = \frac{2}{Z_n + 1}$$

$$Z_2 = \frac{2}{4+1} = \frac{2}{5}$$

$$Z_3 = \frac{10}{7} \quad \text{(from part (i))}$$

$$Z_4 = \frac{2}{\left(\frac{10}{7}+1\right)} \times \frac{7}{7} = \frac{14}{10+7} = \frac{14}{17}$$

$$Z_5 = \frac{2}{\left(\frac{14}{17}+1\right)} \times \frac{17}{17} = \frac{34}{14+17} = \frac{34}{31}$$

$$Z_6 = \frac{2}{\left(\frac{34}{31}+1\right)} \times \frac{31}{31} = \frac{62}{34+31} = \frac{62}{65}$$

Z_1	Z_2	Z_3	Z_4	Z_5	Z_6
4	$\frac{2}{5}$	$\frac{10}{7}$	$\frac{14}{17}$	$\frac{34}{31}$	$\frac{62}{65}$

[7 marks]

Lesson 8

GCSE Mathematics
Iteration

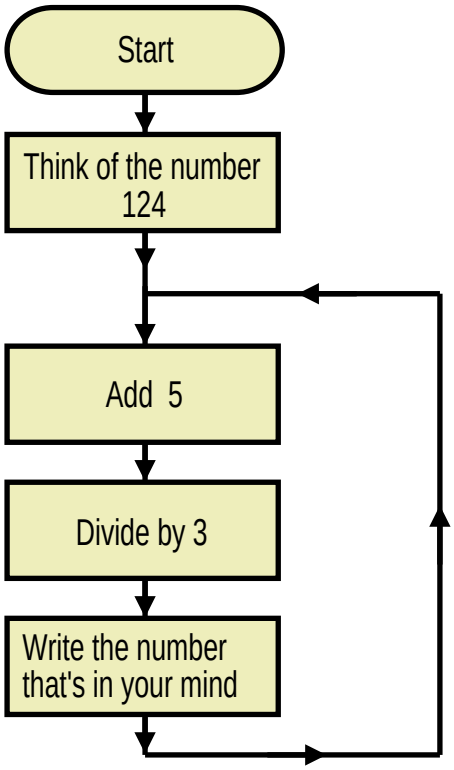
8.1 Homework (Revision)

Non-Calculator

Marks Available : 60

Question 1

A number sequence, *U*, is described by the following flowchart,



Complete this table to show the first six terms in sequence *U*

U_1	U_2	U_3	U_4	U_5	U_6
124					

[7 marks]

Question 2

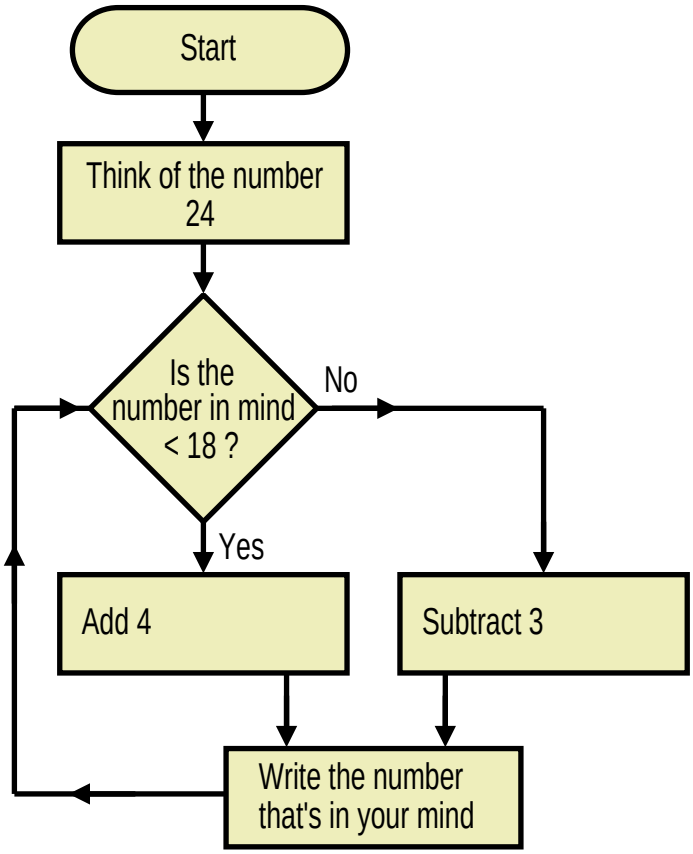
Simplify,

(i) $4 \times \left(\frac{3}{4} + 1 \right)$ **(ii)** $\left(5 + \frac{3}{7} \right) \times 7$

[4 marks]

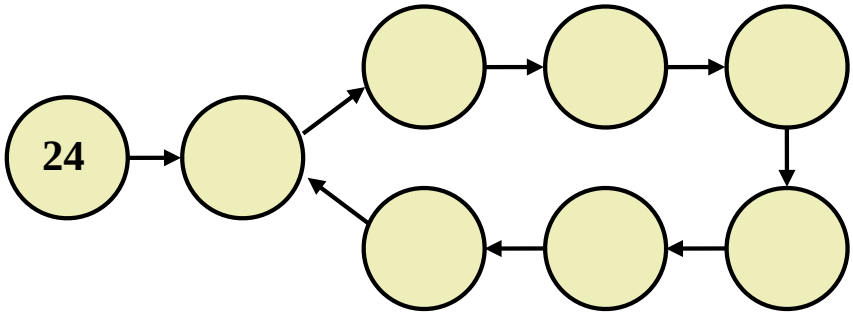
Question 3

A number sequence, *K*, is described by the following flowchart,



The flowchart generates a loop of numbers.

On the following diagram write out the numbers that are in the loop.



[8 marks]

Question 4

First expand the brackets, then simplify,

(i) $\left(4 + \frac{1}{6}\right) \times 6$

[2 marks]

(ii) $\frac{\left(3 + \frac{2}{7}\right)}{4} \times \frac{7}{7}$

[2 marks]

Question 5

A sequence of numbers has the iterative rule

$$A_1 = \frac{1}{16} \qquad A_{n+1} = \frac{4}{3} A_n$$

Use the space below to work out the first six terms of this iterative sequence.
Simplify fractions where possible.
Put your answers in the table.

A_1	A_2	A_3	A_4	A_5	A_6

[7 marks]

Question 6

Simplify,

(i) $\dfrac{\left(\frac{3}{8} + 2\right)}{5} \times \dfrac{8}{8}$

[2 marks]

(ii) $\dfrac{\left(5 + \frac{7}{4}\right)}{11}$

[2 marks]

(iii) $\dfrac{\left(4 + \frac{2}{3}\right)}{5}$

[2 marks]

Question 7

(i) The following sum has an answer that is a rational number.

That is, a number in the form $\frac{p}{q}$ for integer p and q with $q \neq 0$

Determine what that rational number is.

$$\frac{\left(1 - \frac{1}{2}\right)}{3}$$

[2 marks]

(ii) Consider the iteration, $B_1 = \frac{1}{2}$, $B_{n+1} = \frac{1 - B_n}{3}$

Use the space below to work out the first six terms of this iterative sequence and put your answers in the table towards the bottom of the page.

B_1	B_2	B_3	B_4	B_5	B_6

[7 marks]

Question 8

Simplify,

(i) $\frac{1}{\left(\frac{5}{3} + 4\right)} \times \frac{3}{3}$

[2 marks]

(ii) $\frac{5}{\left(\frac{7}{8} + 2\right)}$

[2 marks]

(iii) $\frac{9}{\left(\frac{7}{5} - 1\right)}$

[2 marks]

Question 9

(i) The following sum has an answer that is a rational number.

That is, a number in the form $\frac{p}{q}$ for integer p and q with $q \neq 0$

Determine what that rational number is.

$$\frac{2}{\left(3 + \frac{2}{3}\right)}$$

[2 marks]

(ii) Consider the iteration, $Z_1 = 0$, $Z_{n+1} = \frac{2}{3 + Z_n}$

Use the space below to work out the first six terms of this iterative sequence and put your answers in the table towards the bottom of the page.

Z_1	Z_2	Z_3	Z_4	Z_5	Z_6

[7 marks]

8.2 Answers to 8.1 Homework (Revision)

GCSE Mathematics
Iteration

Answer 1

$$U_1 = 124, \quad U_{n+1} = \frac{U_n + 5}{3}$$

U_1	U_2	U_3	U_4	U_5	U_6
124	43	16	7	4	3

[7 marks]

Answer 2

(i) $4 \times \left(\frac{3}{4} + 1 \right)$

$= 3 + 4$

$= 7$

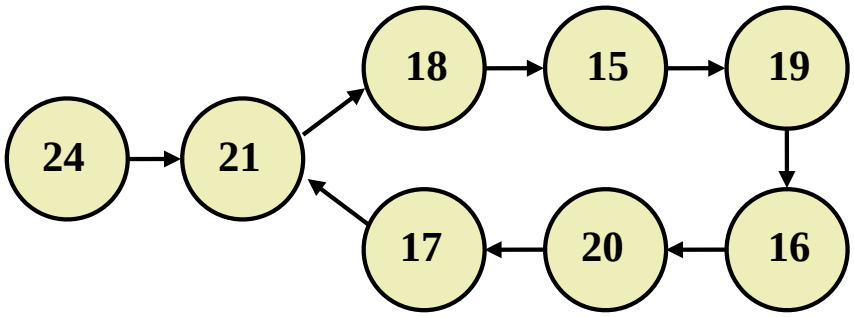
(ii) $\left(5 + \frac{3}{7} \right) \times 7$

$= 35 + 3$

$= 38$

[4 marks]

Answer 3



[8 marks]

Answer 4

(i) $\left(4 + \frac{1}{6} \right) \times 6 = 24 + 1 = 25$

[2 marks]

(ii) $\frac{\left(3 + \frac{2}{7} \right)}{4} \times \frac{7}{7} = \frac{21 + 2}{28} = \frac{23}{28}$

[2 marks]

Answer 5

$A_1 = \frac{1}{16} \qquad A_{n+1} = \frac{4}{3} A_n$

$A_2 = \frac{4}{3} \times \frac{1}{16} = \frac{1}{12}$

$A_3 = \frac{4}{3} \times \frac{1}{12} = \frac{1}{9}$

$A_4 = \frac{4}{3} \times \frac{1}{9} = \frac{4}{27}$

$A_5 = \frac{4}{3} \times \frac{4}{27} = \frac{16}{81}$

$A_6 = \frac{4}{3} \times \frac{16}{81} = \frac{64}{243}$

A_1	A_2	A_3	A_4	A_5	A_6
$\frac{1}{16}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{4}{27}$	$\frac{16}{81}$	$\frac{64}{243}$

[7 marks]

Answer 6

(i) $\frac{\left(\frac{3}{8} + 2\right)}{5} \times \frac{8}{8} = \frac{3 + 16}{40} = \frac{19}{40}$

[2 marks]

(ii) $\frac{\left(5 + \frac{7}{4}\right)}{11} \times \frac{4}{4} = \frac{20 + 7}{44} = \frac{27}{44}$

[2 marks]

(iii) $\frac{\left(4 + \frac{2}{3}\right)}{5} \times \frac{3}{3} = \frac{12 + 2}{15} = \frac{14}{15}$

[2 marks]

Answer 7

(i) $\dfrac{\left(1 - \frac{1}{2}\right)}{3} \times \dfrac{2}{2} = \dfrac{2 - 1}{6} = \dfrac{1}{6}$

[2 marks]

(ii) $B_1 = \dfrac{1}{2}$, $B_{n+1} = \dfrac{1 - B_n}{3}$

$B_2 = \dfrac{1}{6}$ (from part (i))

$B_3 = \dfrac{\left(1 - \frac{1}{6}\right)}{3} \times \dfrac{6}{6} = \dfrac{6 - 1}{18} = \dfrac{5}{18}$

$B_4 = \dfrac{\left(1 - \frac{5}{18}\right)}{3} \times \dfrac{18}{18} = \dfrac{18 - 5}{54} = \dfrac{13}{54}$

$B_5 = \dfrac{\left(1 - \frac{13}{54}\right)}{3} \times \dfrac{54}{54} = \dfrac{54 - 13}{162} = \dfrac{41}{162}$

$B_6 = \dfrac{\left(1 - \frac{41}{162}\right)}{3} \times \dfrac{162}{162} = \dfrac{162 - 41}{486} = \dfrac{121}{486}$

B_1	B_2	B_3	B_4	B_5	B_6
$\frac{1}{2}$	$\frac{1}{6}$	$\frac{5}{18}$	$\frac{13}{54}$	$\frac{41}{162}$	$\frac{121}{486}$

[7 marks]

Answer 8

(i) $\dfrac{1}{\left(\frac{5}{3} + 4\right)} \times \dfrac{3}{3} = \dfrac{3}{5 + 12} = \dfrac{3}{17}$

[2 marks]

(ii) $\dfrac{5}{\left(\frac{7}{8} + 2\right)} \times \dfrac{8}{8} = \dfrac{40}{7 + 16} = \dfrac{40}{23}$

[2 marks]

(iii) $\dfrac{9}{\left(\frac{7}{5} - 1\right)} \times \dfrac{5}{5} = \dfrac{45}{7 - 5} = \dfrac{45}{2}$

[2 marks]

Answer 9

(i) $\dfrac{2}{\left(3 + \frac{2}{3}\right)} \times \dfrac{3}{3} = \dfrac{6}{9 + 2} = \dfrac{6}{11}$

[2 marks]

(ii) $Z_1 = 0,\ Z_{n+1} = \dfrac{2}{3 + Z_n}$

$Z_2 = \dfrac{2}{3 + 0} = \dfrac{2}{3}$

$Z_3 = \dfrac{6}{11}$ (from part (i))

$Z_4 = \dfrac{2}{\left(3 + \frac{6}{11}\right)} \times \dfrac{11}{11} = \dfrac{22}{33 + 6} = \dfrac{22}{39}$

$Z_5 = \dfrac{2}{\left(3 + \frac{22}{39}\right)} \times \dfrac{39}{39} = \dfrac{78}{117 + 22} = \dfrac{78}{139}$

$Z_6 = \dfrac{2}{\left(3 + \frac{78}{139}\right)} \times \dfrac{139}{139} = \dfrac{278}{417 + 78} = \dfrac{278}{495}$

Z_1	Z_2	Z_3	Z_4	Z_5	Z_6
0	$\frac{2}{3}$	$\frac{6}{11}$	$\frac{22}{39}$	$\frac{78}{139}$	$\frac{278}{495}$

[7 marks]

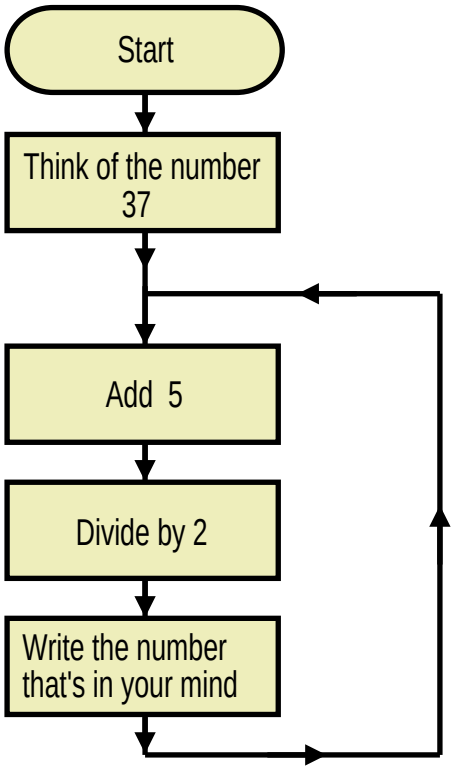
Lesson 9

GCSE Mathematics
Iteration

9.1 Test

Non-Calculator
Marks Available : 60

Question 1
A number sequence, *U*, is described by the following flowchart,



Complete this table to show the first six terms in sequence *U*

U_1	U_2	U_3	U_4	U_5	U_6
37					

[7 marks]

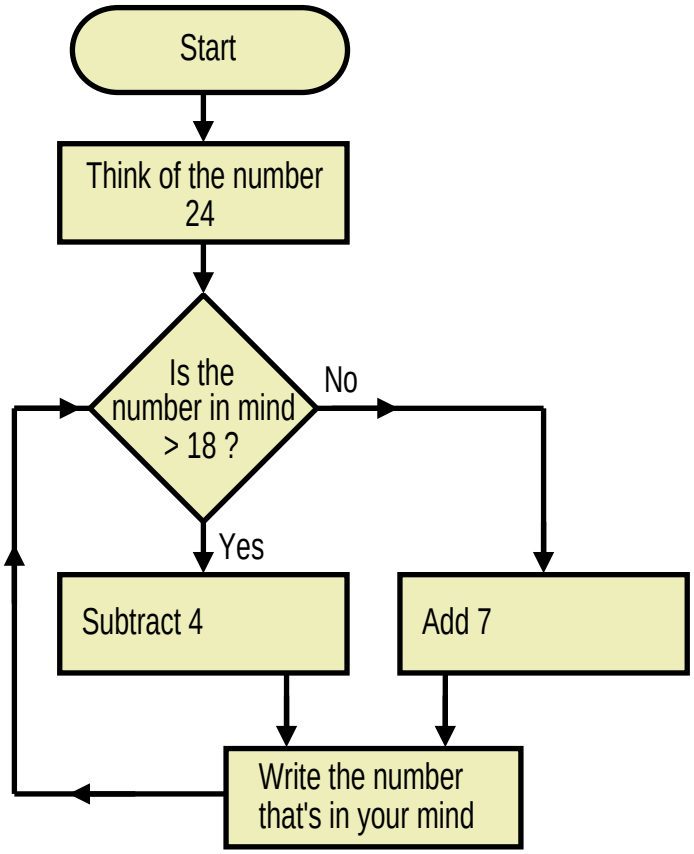
Question 2
Simplify,

(i) $11 \times \left(\frac{5}{11} + 1 \right)$ **(ii)** $\left(6 + \frac{2}{5} \right) \times 5$

[4 marks]

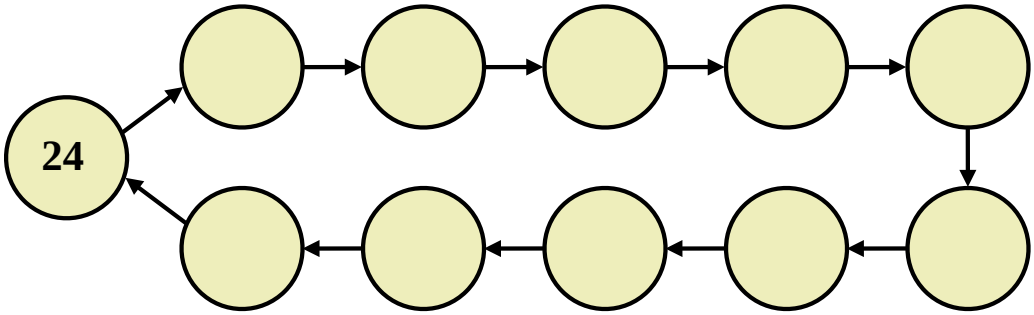
Question 3

A number sequence, *K*, is described by the following flowchart,



The flowchart generates a loop of numbers.

On the following diagram write out the numbers that are in the loop.



[8 marks]

Question 4

First expand the brackets, then simplify,

(i) $\left(2 + \frac{5}{6}\right) \times 6$

[2 marks]

(ii) $\frac{\left(4 + \frac{3}{7}\right)}{1} \times \frac{7}{7}$

[2 marks]

Question 5

A sequence of numbers has the iterative rule

$$A_1 = \frac{1}{8} \qquad A_{n+1} = \frac{2}{3} A_n$$

Use the space below to work out the first six terms of this iterative sequence.
Simplify fractions where possible.
Put your answers in the table.

A_1	A_2	A_3	A_4	A_5	A_6

[7 marks]

Question 6

Simplify,

(i) $\dfrac{\left(\frac{3}{10} + 2\right)}{5} \times \dfrac{10}{10}$

[2 marks]

(ii) $\dfrac{\left(4 + \frac{7}{5}\right)}{11}$

[2 marks]

(iii) $\dfrac{\left(8 + \frac{5}{3}\right)}{4}$

[2 marks]

Question 7

(i) The following sum has an answer that is a rational number.

That is, a number in the form $\frac{p}{q}$ for integer p and q with $q \neq 0$

Determine what that rational number is.

$$\frac{\left(5 - \frac{3}{2}\right)}{2}$$

[2 marks]

(ii) Consider the iteration, $B_1 = 1$, $B_{n+1} = \frac{5 - B_n}{2}$

Use the space below to work out the first six terms of this iterative sequence and put your answers in the table towards the bottom of the page.

B_1	B_2	B_3	B_4	B_5	B_6

[7 marks]

Question 8

Simplify,

(i) $\frac{1}{\left(\frac{5}{6} + 2\right)} \times \frac{6}{6}$

[2 marks]

(ii) $\frac{3}{\left(\frac{7}{9} + 2\right)}$

[2 marks]

(iii) $\frac{11}{\left(\frac{7}{3} - 1\right)}$

[2 marks]

Question 9

(i) The following sum has an answer that is a rational number.

That is, a number in the form $\frac{p}{q}$ for integer p and q with $q \neq 0$

Determine what that rational number is.

$$\frac{1}{\left(2 + \frac{1}{3}\right)}$$

[2 marks]

(ii) Consider the iteration, $Z_1 = -1,\ Z_{n+1} = \frac{1}{2 + Z_n}$

Use the space below to work out the first six terms of this iterative sequence and put your answers in the table towards the bottom of the page.

Z_1	Z_2	Z_3	Z_4	Z_5	Z_6

[7 marks]

9.2 Answers to 9.1 Test

GCSE Mathematics
Iteration

Answer 1

$$U_1 = 37, \quad U_{n+1} = \frac{U_n + 5}{2}$$

U_1	U_2	U_3	U_4	U_5	U_6
37	21	13	9	7	6

[7 marks]

Answer 2

(i) $11 \times \left(\frac{5}{11} + 1 \right)$

$= 5 + 11$

$= 16$

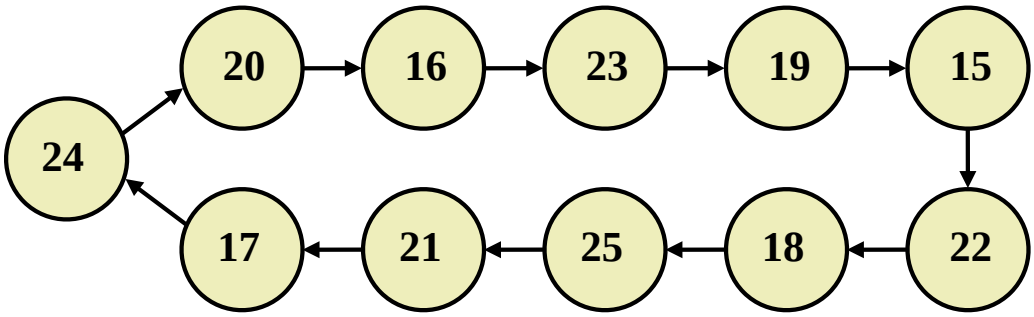
(ii) $\left(6 + \frac{2}{5} \right) \times 5$

$= 30 + 2$

$= 32$

[4 marks]

Answer 3



[8 marks]

Answer 4

(i) $\left(2 + \frac{5}{6} \right) \times 6 = 12 + 5 = 17$

[2 marks]

(ii) $\frac{\left(4 + \frac{3}{7} \right)}{1} \times \frac{7}{7} = \frac{28 + 3}{7} = \frac{31}{7}$

[2 marks]

Answer 5

$A_1 = \frac{1}{8} \qquad A_{n+1} = \frac{2}{3} A_n$

$A_2 = \frac{2}{3} \times \frac{1}{8} = \frac{1}{12}$

$A_3 = \frac{2}{3} \times \frac{1}{12} = \frac{1}{18}$

$A_4 = \frac{2}{3} \times \frac{1}{18} = \frac{1}{27}$

$A_5 = \frac{2}{3} \times \frac{1}{27} = \frac{2}{81}$

$A_6 = \frac{2}{3} \times \frac{2}{81} = \frac{4}{243}$

A_1	A_2	A_3	A_4	A_5	A_6
$\frac{1}{8}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{27}$	$\frac{2}{81}$	$\frac{4}{243}$

[7 marks]

Answer 6

(i) $\frac{\left(\frac{3}{10} + 2\right)}{5} \times \frac{10}{10} = \frac{3 + 20}{50} = \frac{23}{50}$

[2 marks]

(ii) $\frac{\left(4 + \frac{7}{5}\right)}{11} \times \frac{5}{5} = \frac{20 + 7}{55} = \frac{27}{55}$

[2 marks]

(iii) $\frac{\left(8 + \frac{5}{3}\right)}{4} \times \frac{3}{3} = \frac{24 + 5}{12} = \frac{29}{12}$

[2 marks]

Answer 7

(i) $\dfrac{\left(5 - \frac{3}{2}\right)}{2} \times \dfrac{2}{2} = \dfrac{10 - 3}{4} = \dfrac{7}{4}$

[2 marks]

(ii) $B_1 = 1,\ B_{n+1} = \dfrac{5 - B_n}{2}$

$B_2 = \dfrac{5 - 1}{2} = \dfrac{4}{2} = 2$

$B_3 = \dfrac{5 - 2}{2} = \dfrac{3}{2}$

$B_4 = \dfrac{7}{4}$ (from part (i))

$B_5 = \dfrac{\left(5 - \frac{7}{4}\right)}{2} \times \dfrac{4}{4} = \dfrac{20 - 7}{8} = \dfrac{13}{8}$

$B_6 = \dfrac{\left(5 - \frac{13}{8}\right)}{2} \times \dfrac{8}{8} = \dfrac{40 - 13}{16} = \dfrac{27}{16}$

B_1	B_2	B_3	B_4	B_5	B_6
1	2	$\frac{3}{2}$	$\frac{7}{4}$	$\frac{13}{8}$	$\frac{27}{16}$

[7 marks]

Answer 8

(i) $\dfrac{1}{\left(\frac{5}{6} + 2\right)} \times \dfrac{6}{6} = \dfrac{6}{5 + 12} = \dfrac{6}{17}$

[2 marks]

(ii) $\dfrac{3}{\left(\frac{7}{9} + 2\right)} \times \dfrac{9}{9} = \dfrac{27}{7 + 18} = \dfrac{27}{25}$

[2 marks]

(iii) $\dfrac{11}{\left(\frac{7}{3} - 1\right)} \times \dfrac{3}{3} = \dfrac{33}{7 - 3} = \dfrac{33}{4}$

[2 marks]

Answer 9

(i) $\dfrac{1}{\left(2 + \frac{1}{3}\right)} \times \dfrac{3}{3} = \dfrac{3}{6 + 1} = \dfrac{3}{7}$

[2 marks]

(ii) $Z_1 = -1,\ Z_{n+1} = \dfrac{1}{2 + Z_n}$

$Z_2 = \dfrac{1}{2 + (-1)} = 1$

$Z_3 = \dfrac{1}{2 + 1} = \dfrac{1}{3}$

$Z_4 = \dfrac{3}{7}$ (from part (i))

$Z_5 = \dfrac{1}{\left(2 + \frac{3}{7}\right)} \times \dfrac{7}{7} = \dfrac{7}{14 + 3} = \dfrac{7}{17}$

$Z_6 = \dfrac{1}{\left(2 + \frac{7}{17}\right)} \times \dfrac{17}{17} = \dfrac{17}{34 + 7} = \dfrac{17}{41}$

Z_1	Z_2	Z_3	Z_4	Z_5	Z_6
-1	1	$\frac{1}{3}$	$\frac{3}{7}$	$\frac{7}{17}$	$\frac{17}{41}$

[7 marks]