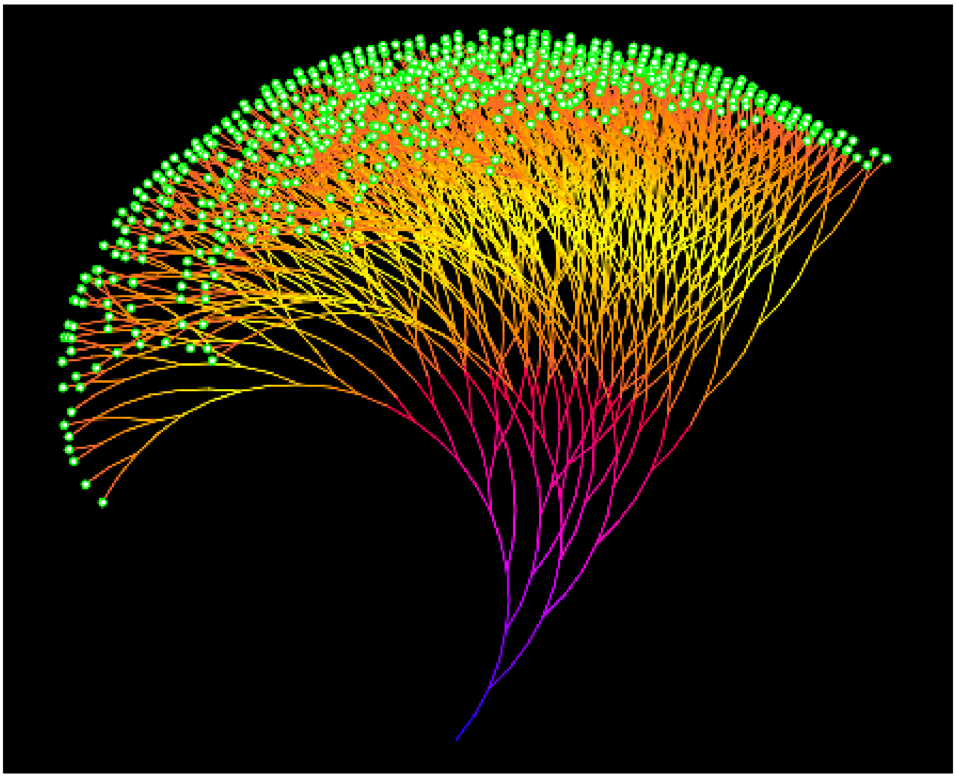
Academic Perspectives

THE COLLATZ CONJECTURE

The easiest to understand unsolved problem in mathematics

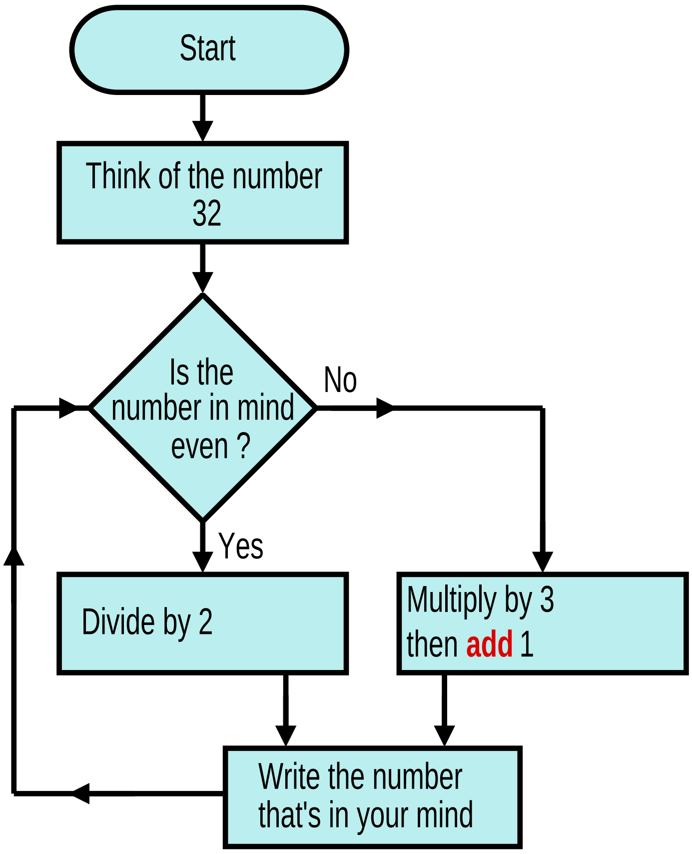


1.1 An Introduction To The Collatz Conjecture

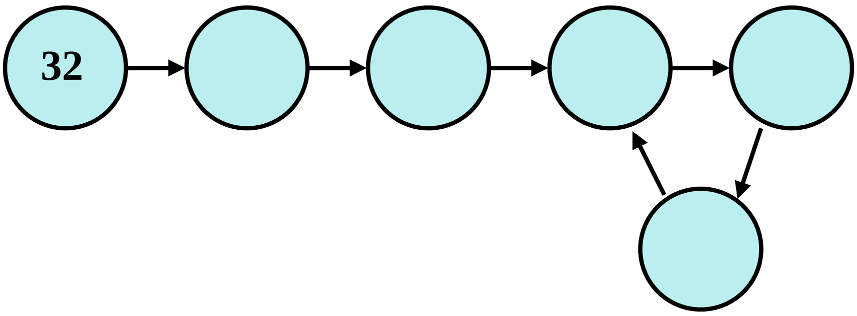
Is the image is of coral seaweed ? Or blood vessels ? Or the neural network of some small creature ? In fact it's a beautiful representation of a sequence of numbers generated from the Collatz conjecture. A conjecture is a mathematical statement that is believed by the person making it to be true, but it may not be. It's an invitation to look into something that another mathematician considers interesting. The conjecture is named after the German mathematician Lothar Collatz (1910-1990) who formulated it in 1937. Simple to state and understand, to this day not known if the Collatz conjecture is true or false. The reputation of the conjecture is such that if you were to settle the matter, one way or the other, you would immediately become famous, not just in the world of mathematics, but across front pages of newspapers around the globe. We are going to take a look at the conjecture, and others like it, and perhaps get to the point where we can produce interesting and colourful representations of the number sequences involved, like the one above.

1.2 The Iteration At The Heart of the Collatz Conjecture

An iterative process is one in which you repeatedly carry out the same set of instructions. The idea of iteration has been around for over two hundred years but only since the invention of the desktop computer in the late 1970s has it become mainstream mathematics. A flowchart can be used to describe an iteration.



On the following diagram write out the numbers generated by the flowchart.



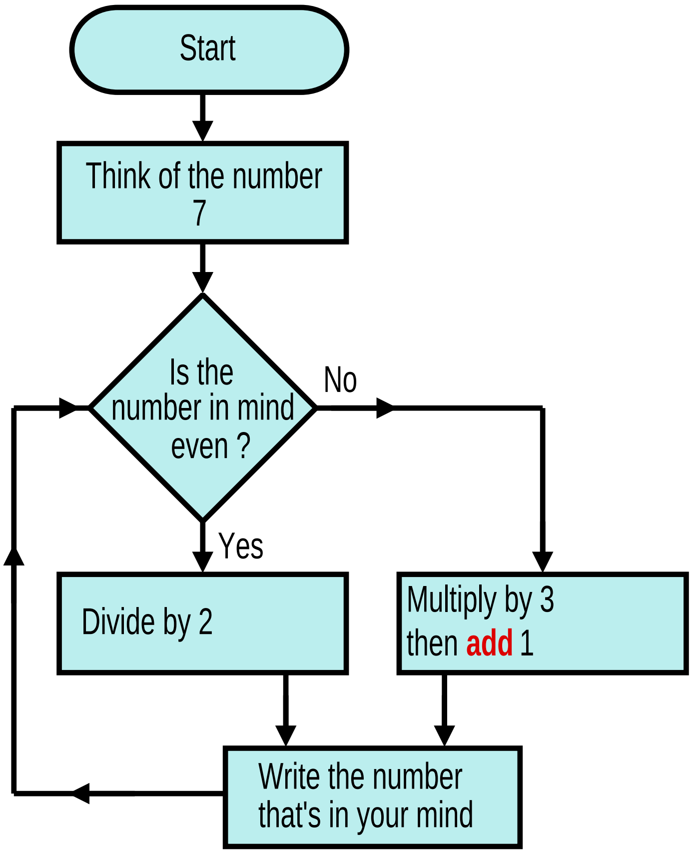
[2 marks]

The iteration described in the flowchart shows that, if you start with the number 32, the sequence eventually reaches the number 1. You could, of course, start with a different positive integer. Would the same iteration rule, starting from this different number, also pass through the number 1 ?

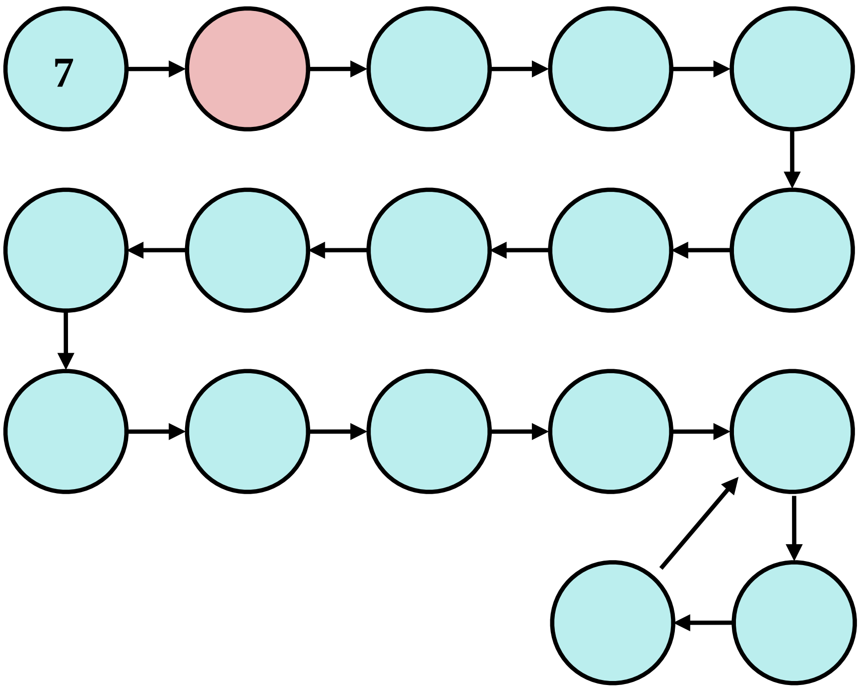
Let's begin to investigate by starting with the number 7.

(The 7 is my choice; you'll get to chose your own starting number soon enough!)

1.3 Starting From Seven



On the following diagram write out the numbers generated starting from 7.



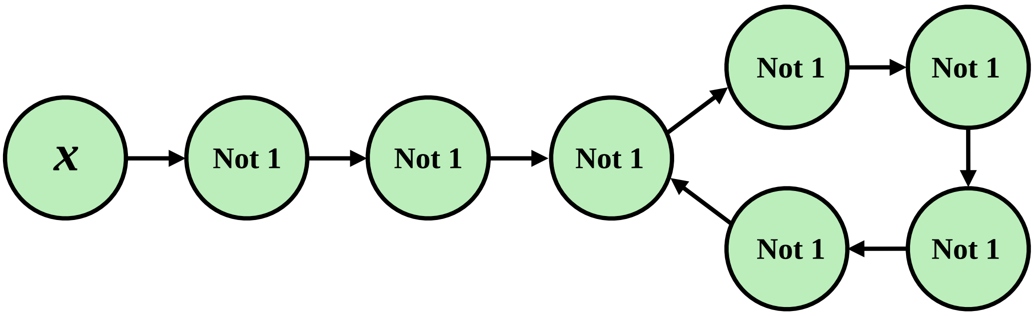
[3 marks]

1.4 The Collatz Conjecture

The Collatz Conjecture claims that, ***no matter what positive integer you start with***, the resulting number sequence will always eventually pass through the number one. Some sequences, like that starting from seven, increase and decrease many times but the claim is that, all the sequences will “fall to earth”, meaning that eventually the number one will occur. To this day no one has found a counterexample of a starting number for which this does not happen, but nor can anyone find a logical explanation for why it will always happen.

1.5 The Search For A Counterexample

The most straight forward way to prove that the Collatz conjecture is false is to find a counterexample. The counterexample would be a starting number that perhaps causes the terms in the resulting sequence to increase and decrease several times before going into a loop, all without involving the number one. So, for example, starting with *x*, the number path could look something like this;



Generally speaking, when faced with a conjecture, it's always worth initially trying to find a counterexample. Trying to prove that something is true, when in fact it is false, is a good way to waste a lot of time. This is because mathematicians are primarily interested in showing that conjectures are true. The following joke illustrate this;

My Mathematical Week

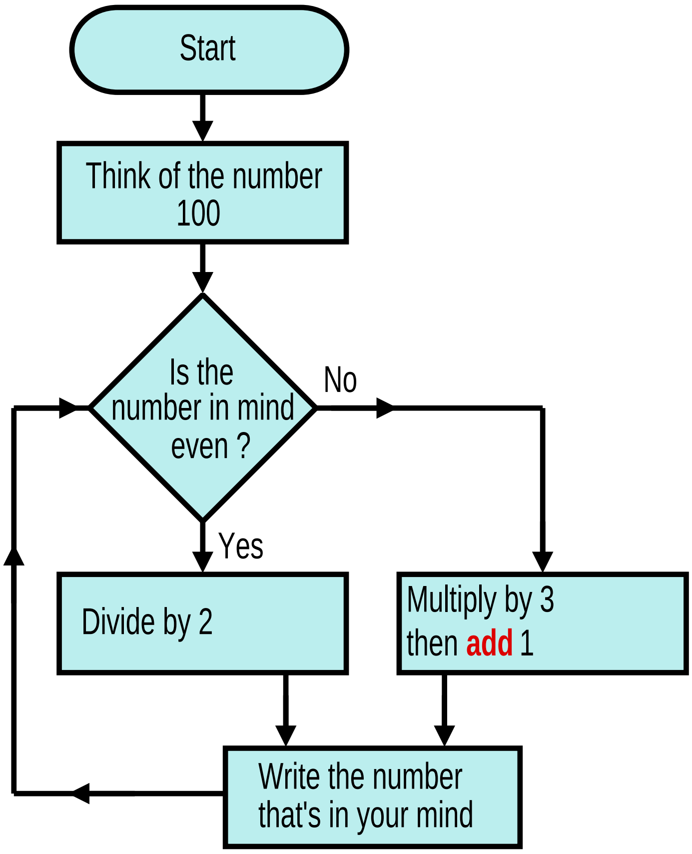
Monday: tried to prove my conjecture is true
Tuesday: tried to prove my conjecture is true
Wednesday: tried to prove my conjecture is true
Thursday: tried to prove my conjecture is true
Friday: realised that my conjecture is false

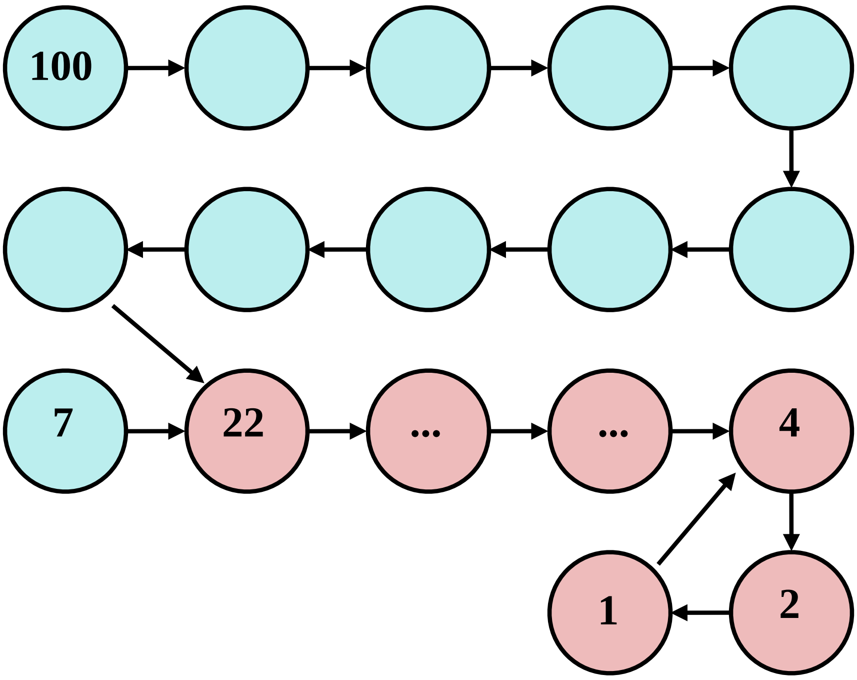
1.6 An Efficient Counterexample Search

Although you are, no doubt, keen to get started on trying out your own starting number, let's look at a situation that will illustrate how we can massively increase the efficiency of our search for a counterexample.

The basic idea behind an efficient search is this; From the work just done, you already know what happens to many other starting numbers. For example, in the red circle of the search that started with 7, you have the number 22. This means that you already know what happens if 22 is used as the start number. So, the only numbers worth trying now are numbers that did not occur in any circles of a previous answer.

There is one other useful observation, best illustrated by an example;





[2 marks]

Notice that as soon as any number occurred that could be found in a previous diagram you knew how the story would end. So when the number 22 occurred, there was no need to do any more work; you knew how the path then made its way to the terminating loop of [4, 2, 1].

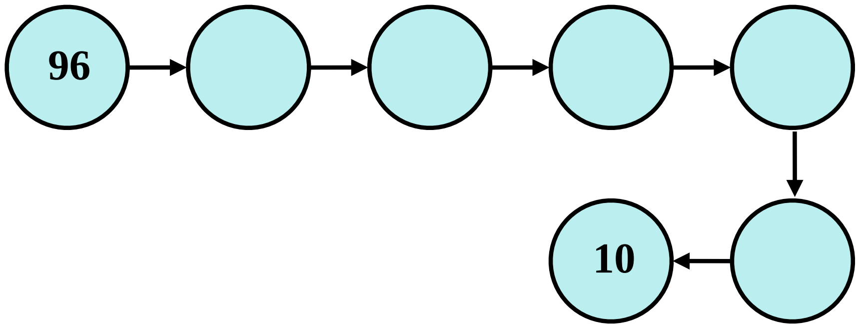
1.7 Exercise

Marks Available : 30

This exercise is about adding a numbers in the correct places in the Collatz tree on the adjacent page and updating the *Collatz 100 Table* at every opportunity.

Question 1

(i) Apply the Collatz iterative rule to the number 96, stopping at 10.



[1 mark]

(ii) Why was there no need to proceed once the number 10 was reached ?

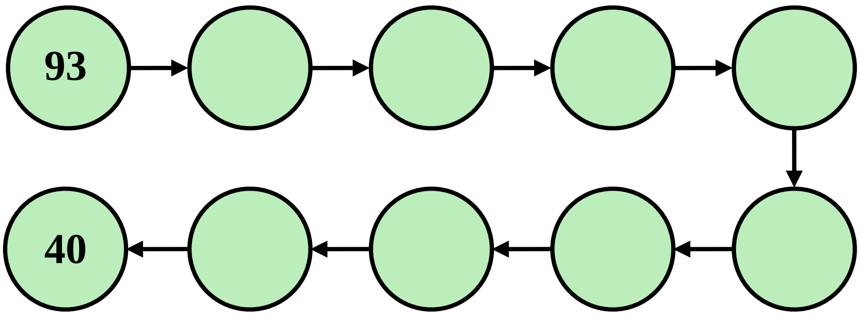
[1 mark]

(iii) Write these on the tree on the adjacent page in the appropriate place.
Update the *Collatz 100 Table* (also on the adjacent page).

[2 marks]

Question 2

(i) Apply the Collatz iterative rule to the number 93, stopping at 40.



[2 marks]

(ii) Write these on the tree on the adjacent page in the appropriate place.
Update the *Collatz 100 Table* (also on the adjacent page).

[2 marks]

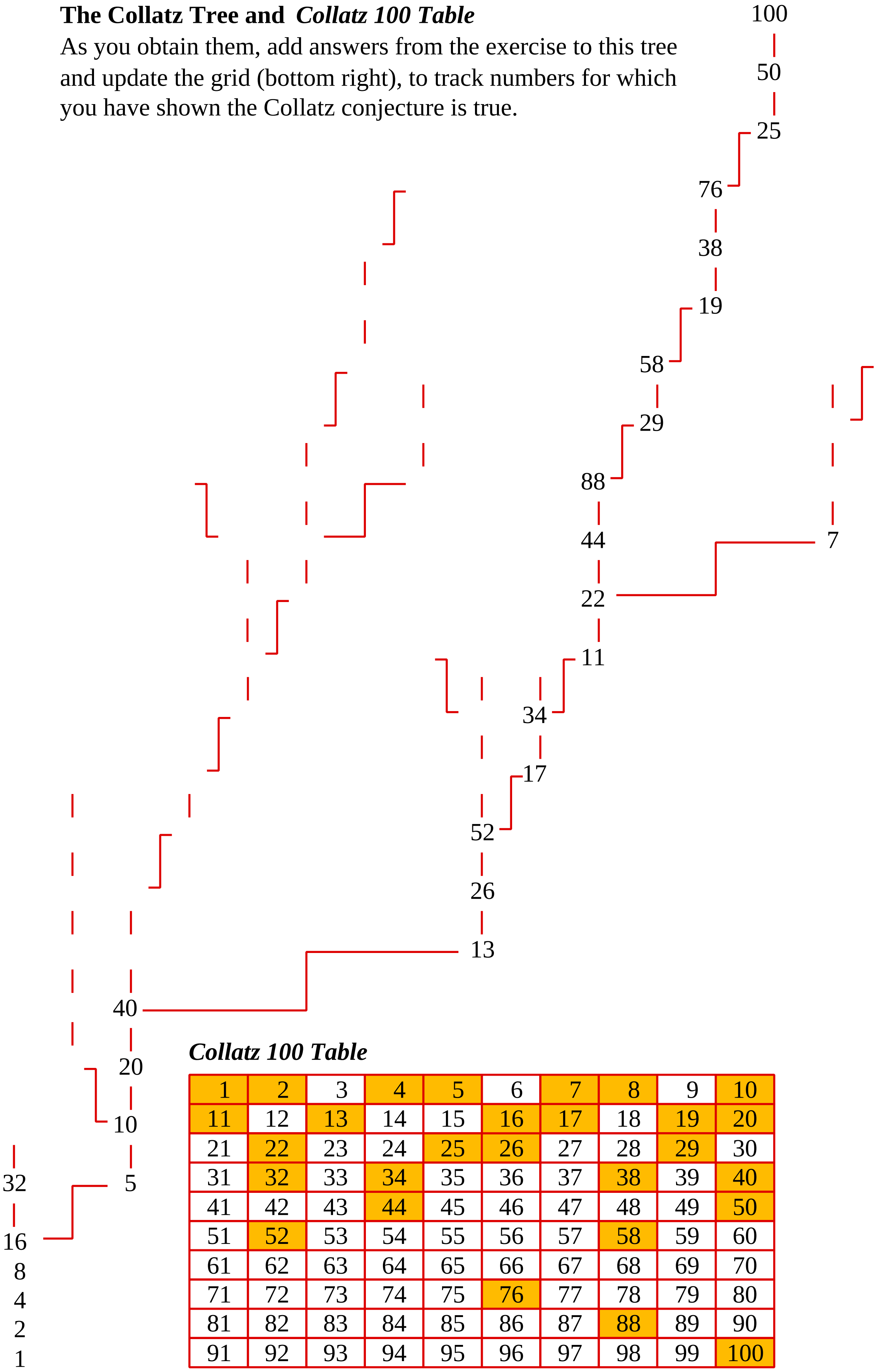
Question 3

One part of working backwards to go up the tree instead of down is easy.
The number directly above any number is double that number.
Work backwards to add a number above 32 on the tree.
Update the *Collatz 100 Table*.

[1 mark]

The Collatz Tree and *Collatz 100 Table*

As you obtain them, add answers from the exercise to this tree and update the grid (bottom right), to track numbers for which you have shown the Collatz conjecture is true.



Collatz 100 Table

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Question 4

(i) Work backwards to add above the 52 on the previous page, the number 104, then, above that, 208.

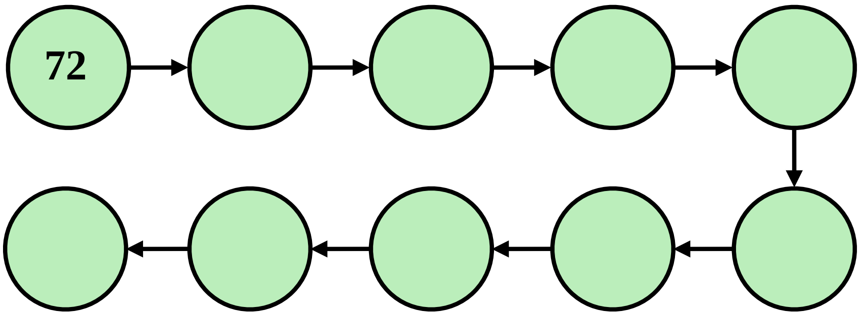
[1 mark]

(ii) To continue going backwards (up) you need to deal with a branch.
We'll look at how to tell if there is a branch or not in the next lesson.
There are two numbers above the 208.
What are these two numbers ?
Write them on the tree on the previous page.
Update the *Collatz 100 Table*.

[2 marks]

Question 5

(i) Come down the tree, starting with 72.
After a short while a number already in the tree is encountered.
There are more circles than you need on the diagram !
(To see if you spot when to stop)



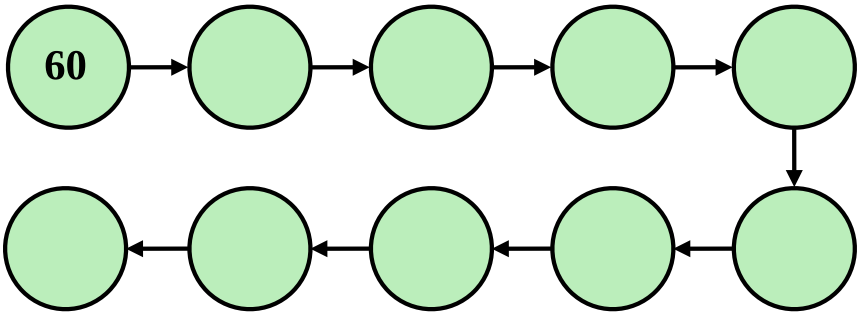
[2 marks]

(ii) Update the tree and the *Collatz 100 table* on the previous page.

[2 marks]

Question 6

(i) Come down the tree, starting with 60.
After a short while a number already in the tree is encountered.
There are more circles than you need on the diagram !
(To see if you spot when to stop)



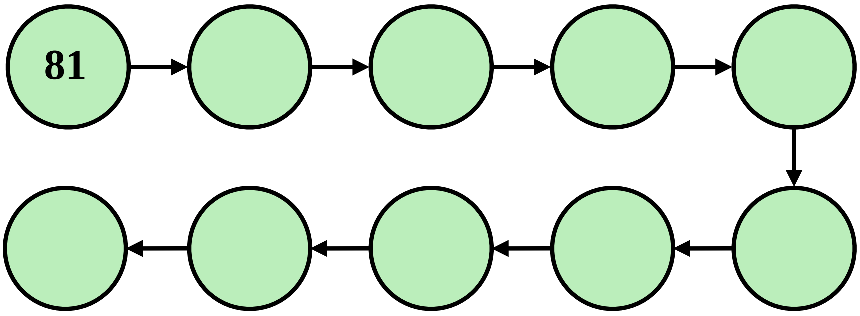
[2 marks]

(ii) Update the tree and the *Collatz 100 table* on the previous page.

[2 marks]

Question 7

(i) Come down the tree, starting with 81.
After a short while a number already in the tree is encountered.
There are more circles than you need on the diagram !
(To see if you spot when to stop)



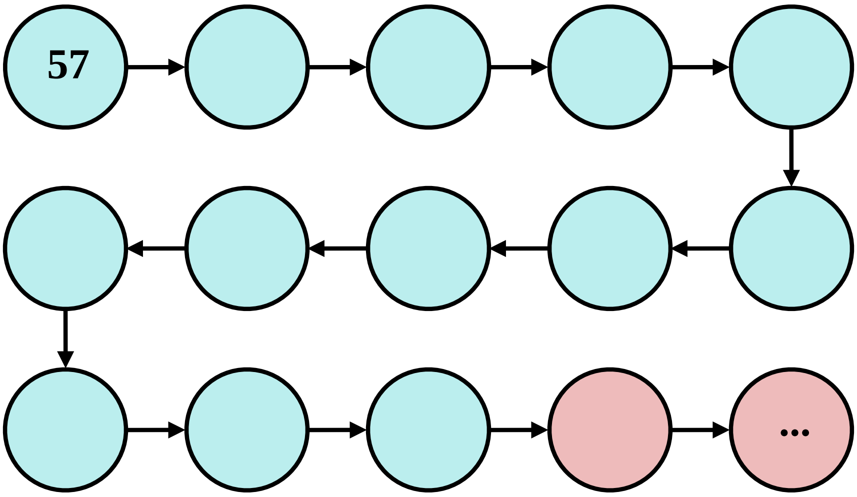
[2 marks]

(ii) Update the tree and the *Collatz 100 table* two pages back.

[2 marks]

Question 8

(i) Come down the tree, starting with 57.
After a while a number already in the tree is encountered.



[3 marks]

(ii) What problem would be encountered if you tried to add this sequence onto the tree we have been building over the last seven questions ?

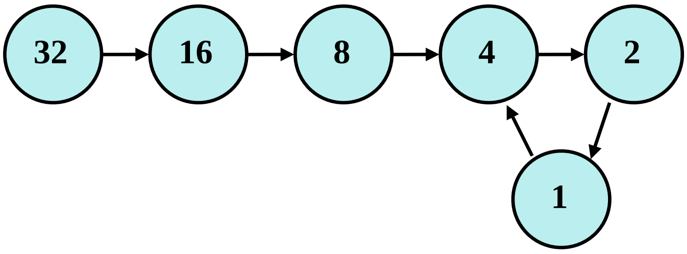
[1 mark]

(iii) This sequence gives you eight more numbers to add to the *Collatz 100 table* of those numbers for which you know the conjecture is true.
Update the *Collatz 100 table* two pages back.

[2 marks]

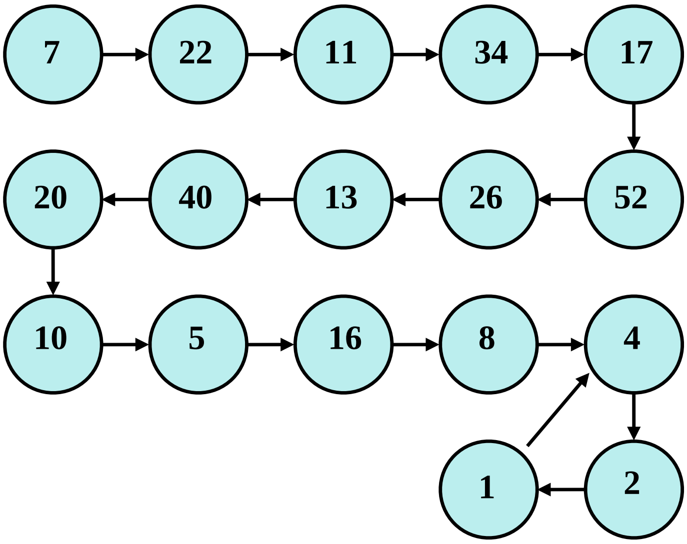
1.8 Answers

1.8.1 Solution to 1.2 The Iteration At The Heart of the Collatz Conjecture



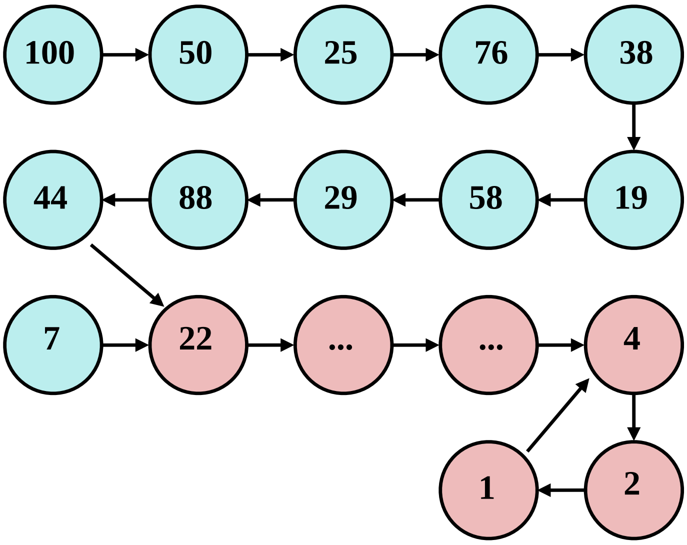
[2 marks]

1.8.2 Solution to 1.3 Starting From Seven



[3 marks]

1.8.3 Solution to 1.6 An Efficient Counterexample Search

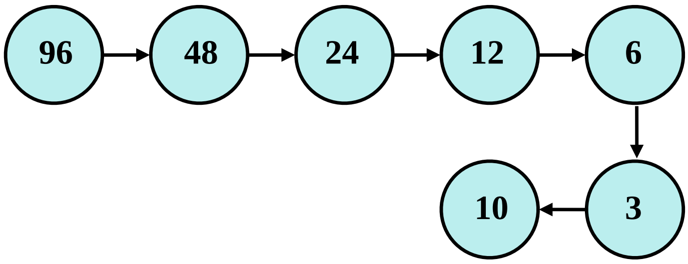


[2 marks]

1.8.4 Solutions to 1.7 Exercise

Answer 1

(i)



[1 mark]

(ii) The number 10 has already occurred (in the starting from 7 sequence) and so, once we got it here we “knew how the story would end”.

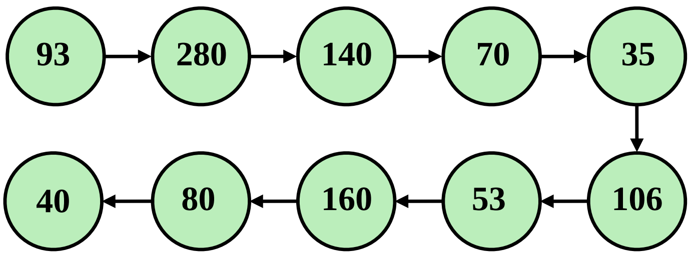
[1 mark]

(iii) See tree and *Collatz 100 table*, two pages further on.

[2 marks]

Answer 2

(i)



[2 marks]

(ii) See tree and *Collatz 100 table*, two pages further on.

[2 marks]

Answer 3

See 64 above 32 on tree and *Collatz 100 table*, two pages further on.

[1 mark]

Answer 4

(i) See 104 and 208 on tree two pages further on.

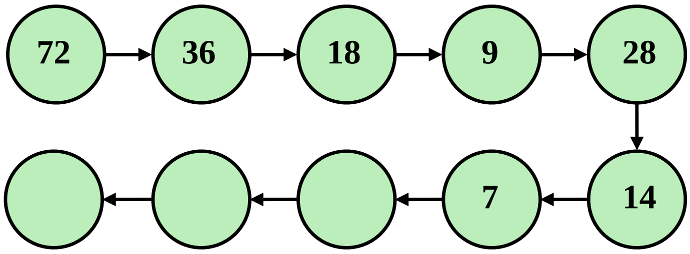
[1 mark]

(ii) See 69 and 416 on tree on the next page.
Add 69 to *Collatz 100 table*, two pages further on.

[2 marks]

Answer 5

(i)



(The 7 occurred in 1.3 Starting From 7)

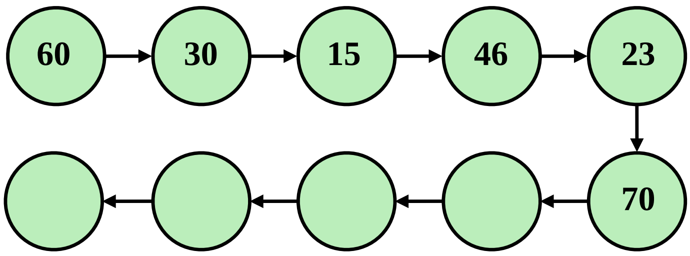
[2 marks]

(ii) See tree and *Collatz 100 table*, on the next page.

[2 marks]

Answer 6

(i)



(The 70 occurred in Answer 2)

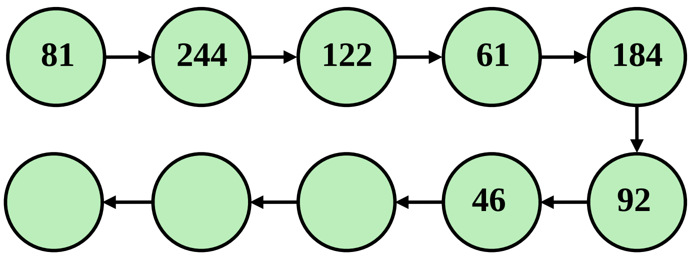
[2 marks]

(ii) See tree and *Collatz 100 table*, on the next page.

[2 marks]

Answer 7

(i)



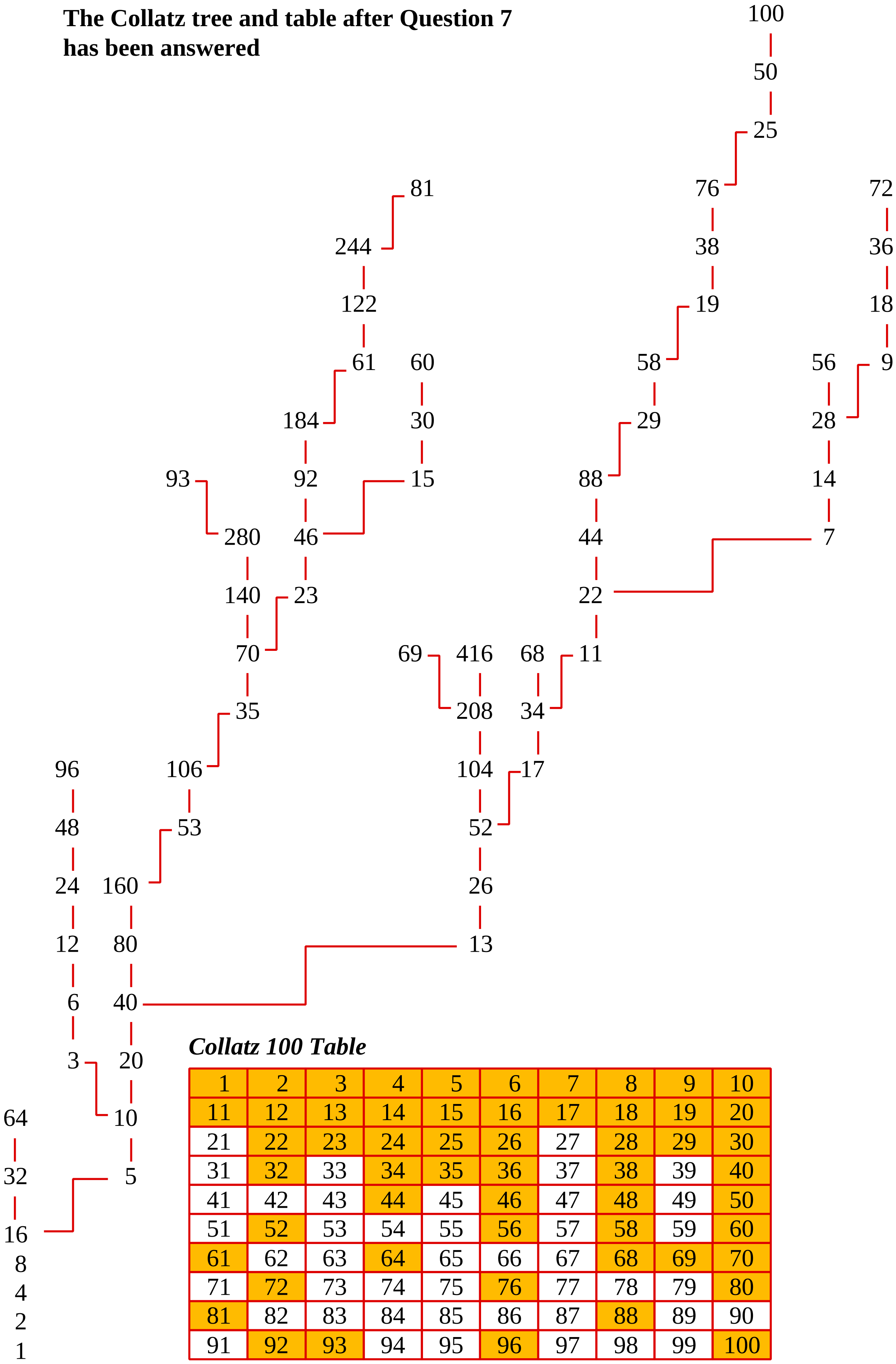
(The 46 occurred in Answer 6)

[2 marks]

(ii) See tree and *Collatz 100 table*, on the next page.

[2 marks]

The Collatz tree and table after Question 7 has been answered



Collatz 100 Table

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
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51	52	53	54	55	56	57	58	59	60
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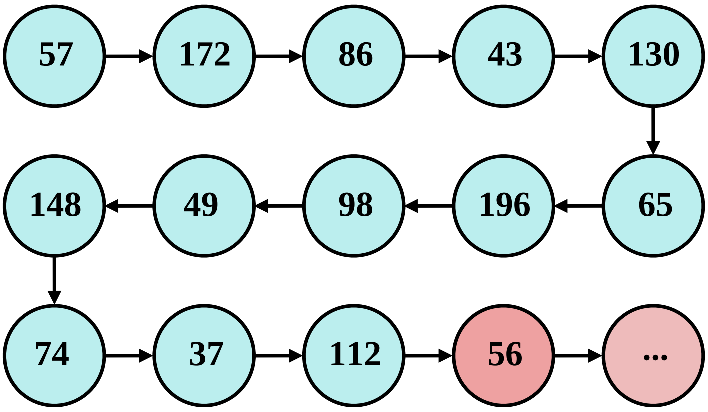
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Answer 8

(i)



[3 marks]

(ii) The extension does not fit onto the paper provided !

[1 mark]

(iii) The eight numbers accounted for by this sequence are shaded yellow in the following table;

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

[2 marks]

2.1 A Mathematical Description

Drawing a flowchart to describe *The Collatz Rule* is time consuming and takes up an excessive amount of space on the page. A more succinct description of the iteration at the heart of the Collatz conjecture is this;

$$y = \begin{cases} \dfrac{x}{2} & \text{if } x \text{ is even} \\ 3x + 1 & \text{if } x \text{ is odd} \end{cases}$$

Our strategy to begin constructing the Collatz tree has been to pick a positive integer that is not in our existing tree and work out its "fall to earth". Here is the *Collatz 100 Table* that we had worked out by the end of the previous lesson,

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
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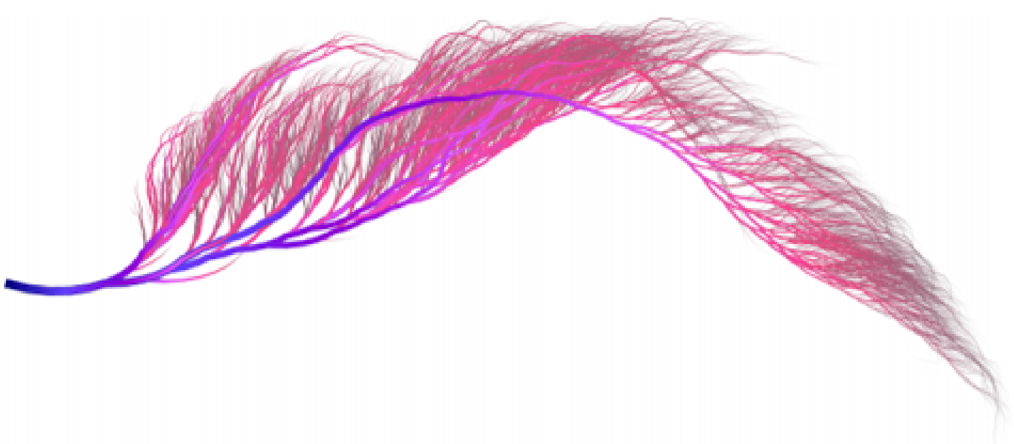
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

The shaded numbers are those for which we know the Collatz conjecture is true.

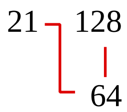
We can be more certain of getting the whole tree by working our way up the tree, starting from the 1 at the base. Computer visualisations of the tree, such as the one below, known as the feather, use this *grow from the base* approach.



An algorithm that will let us move up instead of down is needed. Developing this, which is attended to next, is a little tricky although the end result is pleasingly simple to apply.

2.2 Grow from the Base

The Collatz tree often has a junction where two branches meet. For example, as we have just seen, at the number 64 there is the following;



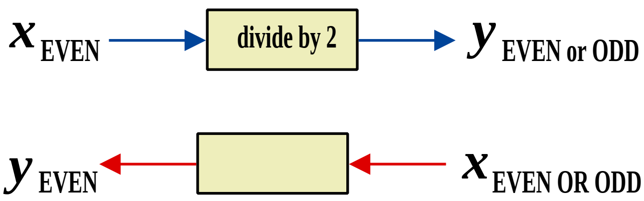
This raises questions such as,

- Do all numbers end up being where two branches meet ?
- Can we, instead of flowing down the tree, flow up ?

Having the mathematical description of the tree is the key to answering these questions. *The Collatz Rule* is,

$$y = \begin{cases} \dfrac{x}{2} & \text{if } x \text{ is even} \\ 3x + 1 & \text{if } x \text{ is odd} \end{cases}$$

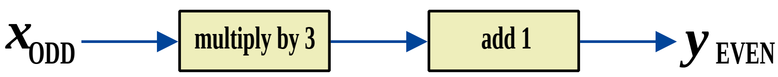
To find a reversing rule for the evens, consider going through the following diagram "the other way";



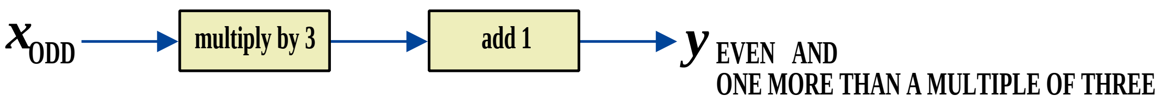
$y_{REVERSE}$ =

[2 marks]

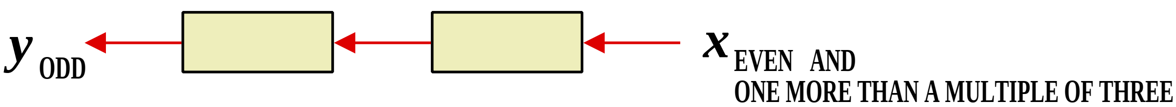
Finding a reversing rule for the odds is more tricky.
To make progress, consider going through the odd rule as before where we multiply by 3 and add 1. What will come out will always be even.



Furthermore, what comes out is one more than a multiple of three.



So, for example, 10 is even and also one more than a multiple of 3 but 8, although even, is not one more than a multiple of three. So when going in reverse it's fine to put 10 in but not 8.



$y_{REVERSE}$ =

[3 marks]

2.3 The Reversed Collatz Rule With Examples

Many animations of the Collatz conjecture start from the number one at the base of the tree and draw the tree going upwards, rather than starting at a randomly chosen starting number the drawing the part of the tree below. *The Reversed Collatz Rule* is the algorithm of how to do this.

The Reversed Collatz Rule

One reverse always occurs and is given by,

$$y_{REVERSE} = 2x$$

Additionally, if x is even and one more than a multiple of three, then there is a branch in the tree and a second reverse which is given by,

$$y_{REVERSE} = \frac{x - 1}{3}$$

Example #1 : Reversing 7

14
|
7

The always occurring reverse is $y_{REVERSE} = 2x = 2 \times 7 = 14$

As 7 is odd there is no second reverse.

Example #2 : Reversing 64

21 ┐ 128
 └ 64

The always occurring reverse is $y_{REVERSE} = 2x = 2 \times 64 = 128$

As 64 is even so ask yourself, is it also one more than a multiple of three ?

It is because 64 divided by 3 has remainder one third (remainder 0.33333...)

There is a second reversal $y_{REVERSE} = \frac{x - 1}{3} = \frac{64 - 1}{3} = \frac{63}{3} = 21$

Example #3 : Reversing 44

88
|
44

The always occurring reverse is $y_{REVERSE} = 2x = 2 \times 44 = 88$

As 44 is even so ask yourself, is it also one more than a multiple of three ?

It is **NOT**: 44 divided by 3 has remainder two thirds (remainder 0.66666...)

There is no second reverse.

2.4 Exercise

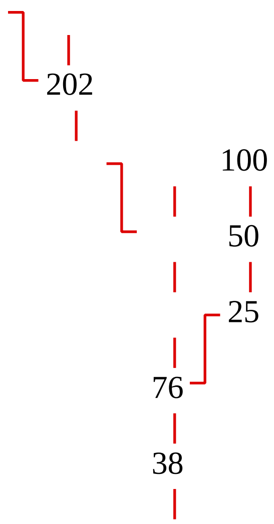
Marks Available : 22 plus 8 bonus

Question 1

(i) Find the two reversals of 202.

[2 marks]

(ii) Use your part (i) answers to help complete the following piece of tree.



[3 marks]

Question 2

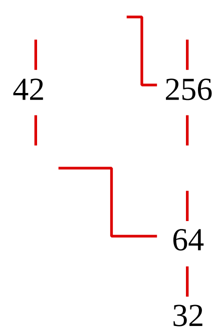
(i) Find all reversals of 42.

[2 marks]

(ii) Find all reversals of 256.

[2 marks]

(iii) Use your part (i) and (ii) answers to help complete this piece of tree;



[3 marks]

Question 3

Mini-Project (15 minutes)

This question is about reversing up the tree beyond the high point reached in this exercise's first question, where 67 was reached.

Starting from 67 begin reversing up the tree.

Explore all branches but stop reversing on a particular branch if any number greater than or equal to 400 occurs.

Draw the tree to illustrate your discoveries.

Update the *Collatz 100 Table* as you progress.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
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51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

[10 marks]

Question 4 (Bonus)

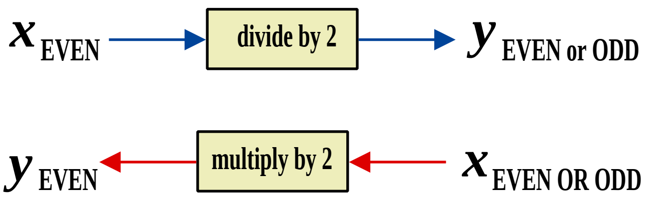
Write a computer program that takes a positive integer as input and writes out the path that number takes as it "falls to earth".

[8 bonus marks]

2.5 Answers

2.5.1 Solutions to 2.2 Grow From The Base

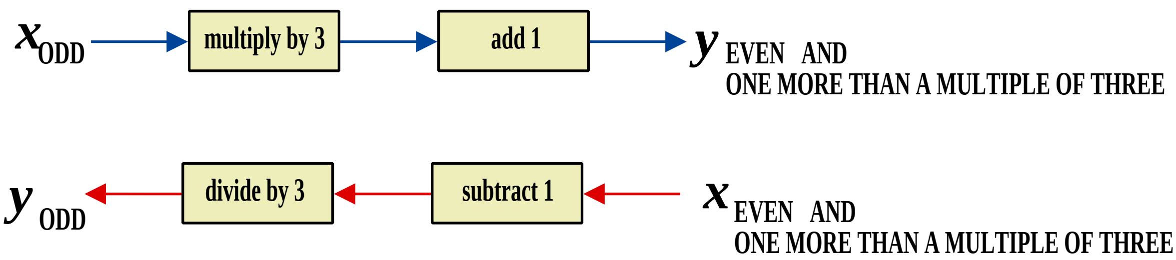
Reversing the rule for when x is even...



$$y_{REVERSE} = 2x$$

[2 marks]

Reversing the rule for when x is odd...

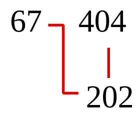


$$y_{REVERSE} = \frac{x - 1}{3}$$

[3 marks]

2.5.2 Solutions to 2.3 Exercise

Answer 1
(i) Reversing 202



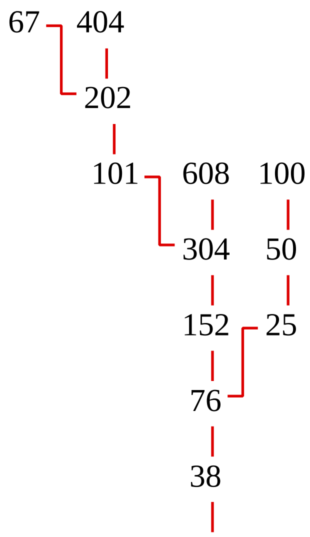
The always occurring reverse is $y_{REVERSE} = 2x = 2 \times 202 = 404$
As 202 is even so ask yourself, is it also one more than a multiple of three ?
It is because 202 divided by 3 has remainder one third (remainder 0.33333...)

There is a second reversal $y_{REVERSE} = \dfrac{x - 1}{3} = \dfrac{202 - 1}{3} = \dfrac{201}{3} = 67$

(This shades 67 in the *Collatz 100 Table*)

[2 marks]

(ii)



[3 marks]

Answer 2

(i) Reversing 42

The always occurring reverse is $y_{REVERSE} = 2x = 2 \times 42 = 84$

As 42 is even so ask yourself, is it also one more than a multiple of three ?

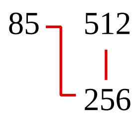
It is **NOT**: 42 divided by 3 has remainder zero thirds.

There is no second reverse.

(This shades 84 in the *Collatz 100 Table*)

[2 marks]

(ii) Reversing 256



The always occurring reverse is $y_{REVERSE} = 2x = 2 \times 256 = 512$

As 256 is even so ask yourself, is it also one more than a multiple of three ?

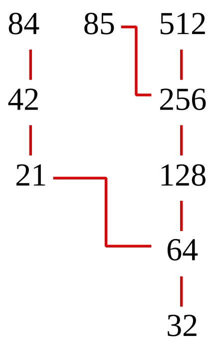
It is because 256 divided by 3 has remainder one third (remainder 0.33333...)

There is a second reversal $y_{REVERSE} = \dfrac{x - 1}{3} = \dfrac{256 - 1}{3} = \dfrac{255}{3} = 85$

(This shades 85 in the *Collatz 100 Table*)

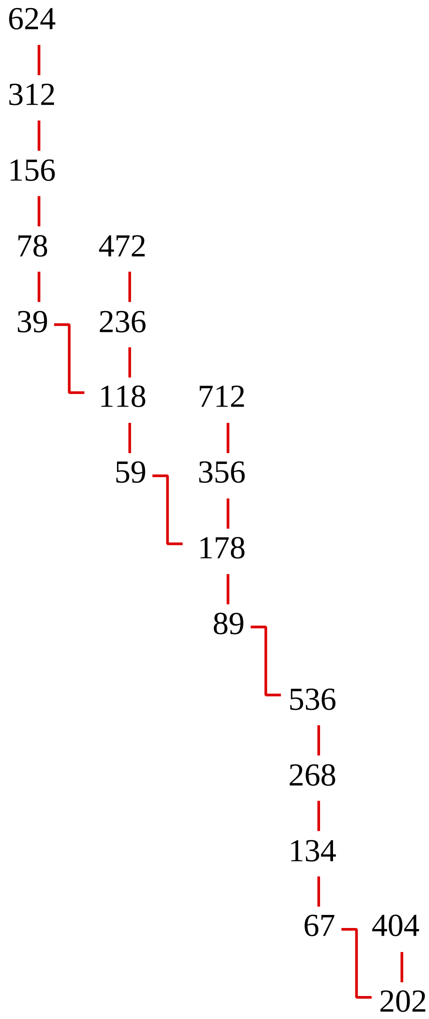
[2 marks]

(iii)



[3 marks]

Answer 3



(This shades 39, 59, 78, and 89 in the *Collatz 100 Table*)

The updated *Collatz 100 Table* is;

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

[10 marks]

Answer 4

Here is a short program in Python
(Written by "Dawg" in 2015 on the stackoverflow website) :

```
print("This is The Collatz Sequence")
user = int(input("Enter a number: "))
def collatz(n):
    print(n)
    while n != 1:
        if n % 2 == 0:
            n = n // 2
            yield(n)
        else:
            n = n * 3 + 1
        yield(n)
print(list(collatz(user)))
```

Sample of output:

```
This is The Collatz Sequence
Enter a number: 3
3
[10, 5, 16, 8, 4, 2, 1]
```

[8 bonus marks]

3.1 The Monster 27 Sequence

Our *Collatz 100 Table* after two lessons looks like this;

Collatz 100 Table

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

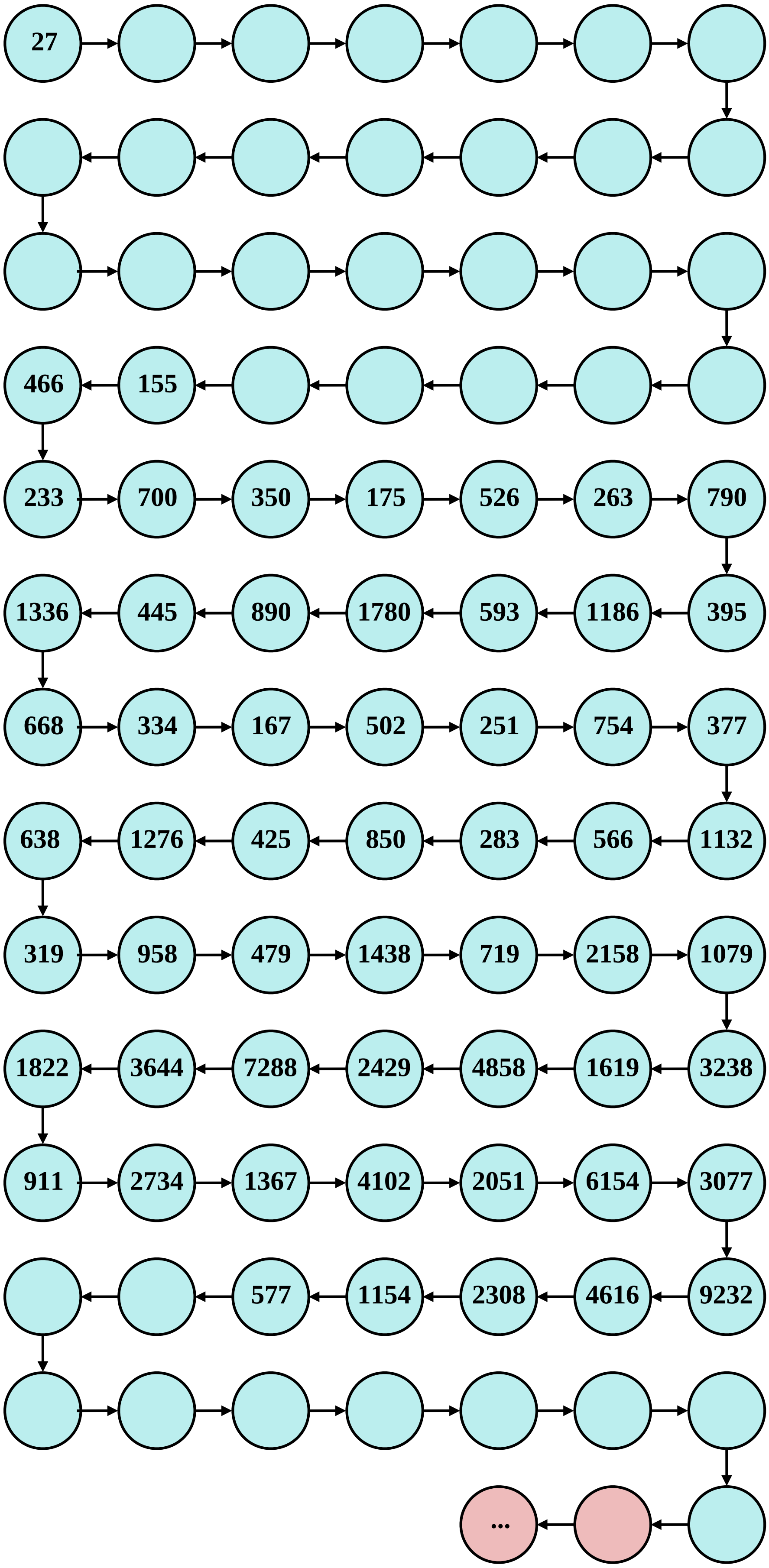
The obvious next number to try is 27.

If you have written your own computer program to generate Collatz sequences use it to obtain the 27 sequence. Otherwise use the online Collatz calculator at;

https://goodcalculators.com/collatz-conjecture-calculator/

Complete the 27 sequence over the page.

- You stop once you get to a number for which you know how the story ends.
- Cross out the appropriate numbers in *Collatz 100 Table* (above).
- Don't forget to also cross out the 27.


27
155
466
233
700
350
175
526
263
790
395
1186
593
1780
890
445
1336
668
334
167
502
251
754
377
1132
566
283
850
425
1276
638
319
958
479
1438
719
2158
1079
3238
1619
4858
2429
7288
3644
1822
911
2734
1367
4102
2051
6154
3077
9232
4616
2308
1154
577
...

[8 marks]

3.2 Exercise

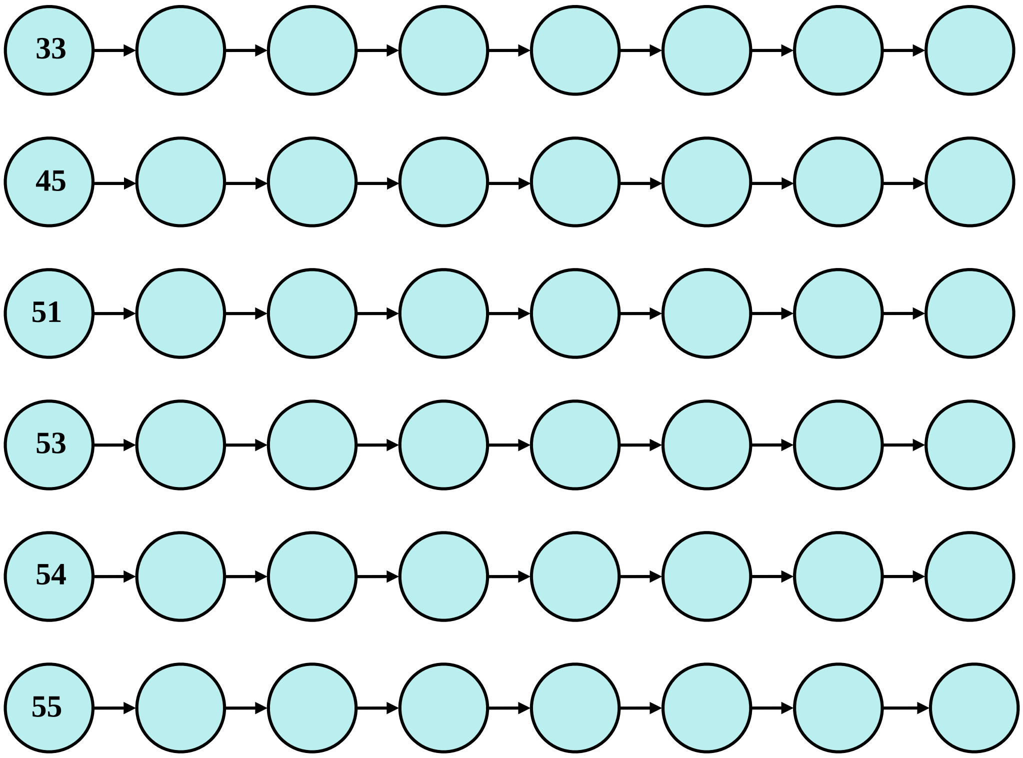
Marks Available : 44

Question 1

Completing the *Collatz 100 Table*

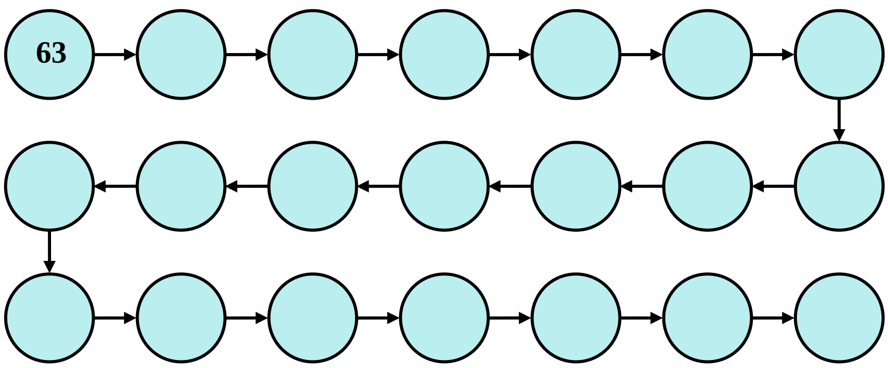
Use a Collatz calculator to systematically check the numbers that have not yet been accounted for in the Collatz 100 Table. Do this with the aid of the following diagrams. You will not need to fill in all of the circles each time because as soon as you get a number that is already accounted for in the table you know how the story ends.

(i) Starting to working through the numbers not yet accounted for...



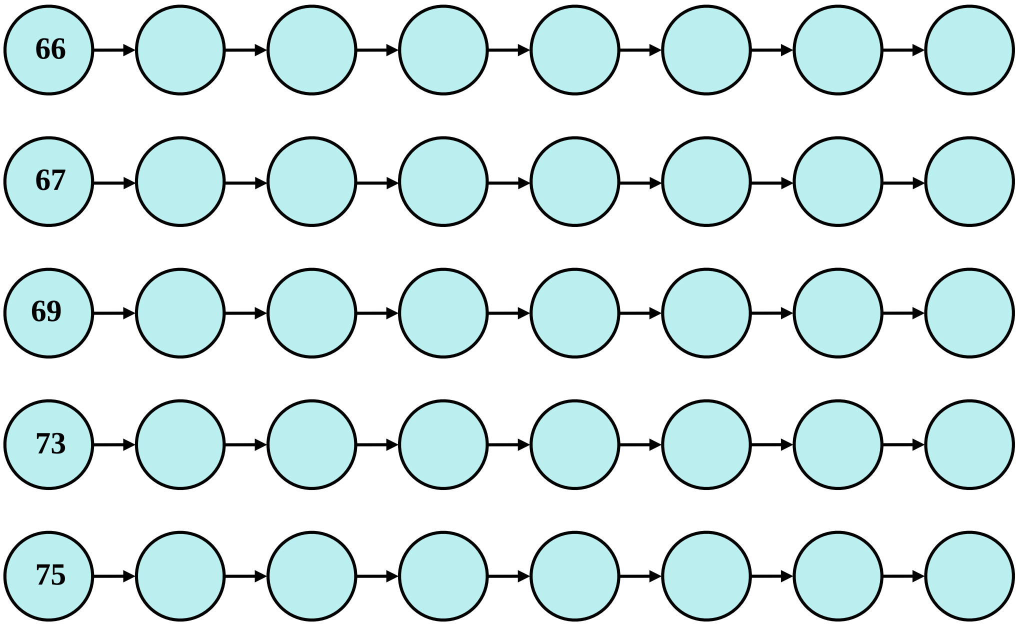
[6 marks]

(ii) A longer sequence...



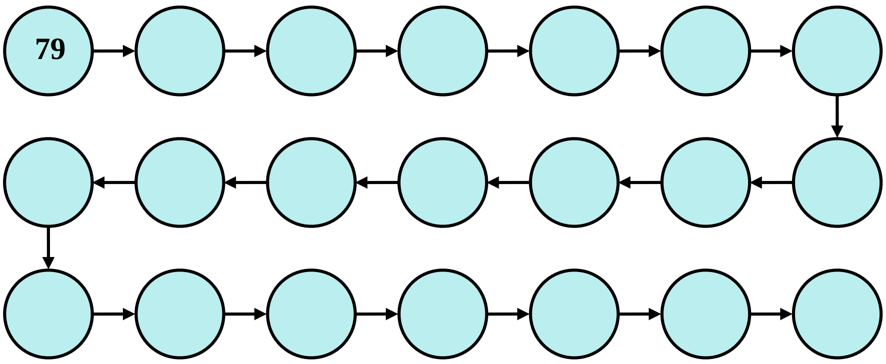
[3 marks]

(iii) Some more that join on to known numbers quickly...



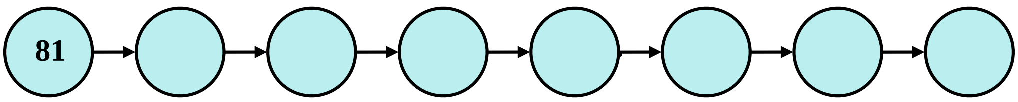
[5 marks]

(iv) Another longer sequence ...



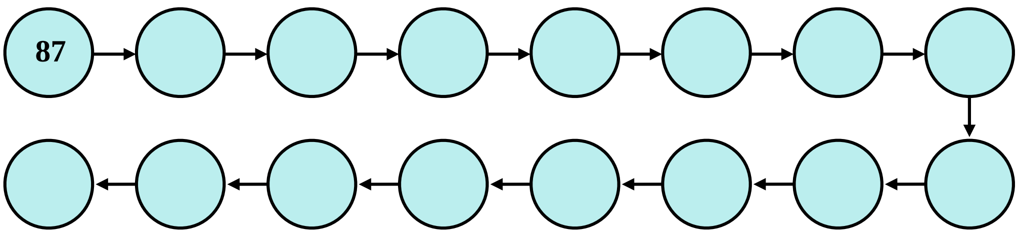
[3 marks]

(v) Short ...



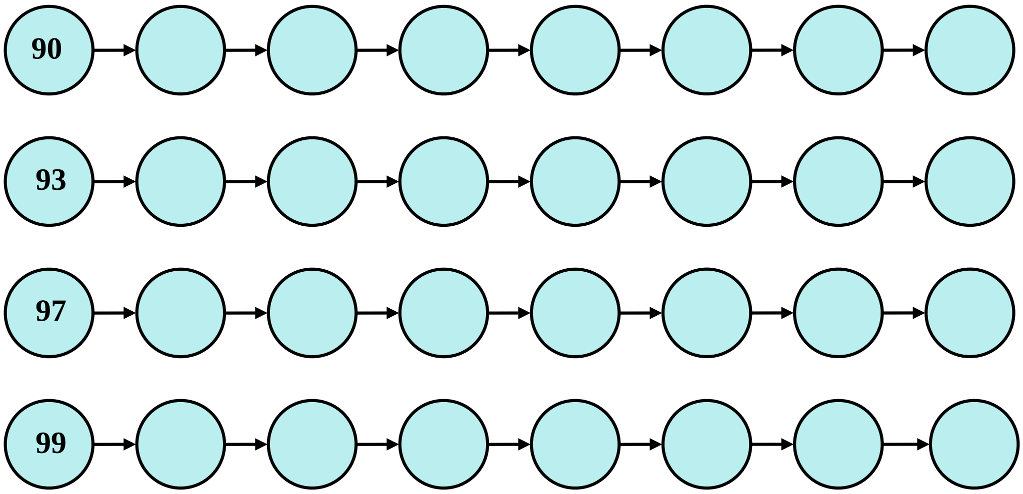
[1 mark]

(vi) Long ...



[2 marks]

(vii) Going for the finish ...



[4 marks]

Question 2

The *Total Stopping Time* of a Collatz sequence is the number of times the starting number is iterated before reaching the number 1.

For example, the number 8 has a *Total Stopping Time* of 3 because from the number 8, the first iteration gives 4, the second iteration gives 2 and the third iteration gives 1.

The online Collatz calculator gives the *Total Stopping Time*.

Here are the *Total Stopping Times* for the *Collatz 100 Table*;

Total Stopping Times 1 - 100

	1	2	3	4	5	6	7	8	9	10
00	0	1	7	2	5	8	16	3	19	6
10	14	9	9	17	17	4	12	20	20	7
20	7	15	15	10	23	10	111	18	18	18
30	106	5	26	13	13	21	21	21	34	8
40	109	8	29	16	16	16	104	11	24	24
50	24	11	11	112	112	19	32	19	32	19
60	19	107	107	6	27	27	27	14	14	14
70	102	22	115	22	14	22	22	35	35	9
80	22	110	110	9	9	30	30	17	30	17
90	92	17	17	105	105	12	118	25	25	25

In the *Total Stopping Times 1- 100* table,

(i) What is the longest stopping time ?

[2 marks]

(ii) Which starting number has this longest stopping time ?

[2 marks]

Amongst mathematicians there is considerable interest in these *Total Stopping Times*. For example, you may have noticed that you often get two and sometimes three different starting numbers next to each other with the same *Total Stopping Time*. This raises the question, can you get four, five, six, ... consecutive starting values with the same *Total Stopping Time* ?

(iii) Complete the following Total Stopping Time table for starting numbers between 101 and 200. You could write a computer program to do this or use the online Collatz calculator.

Total Stopping Times 101 - 200

	1	2	3	4	5	6	7	8	9	10
100										
110										
120										
130										
140										
150										
160										
170										
180										
190										

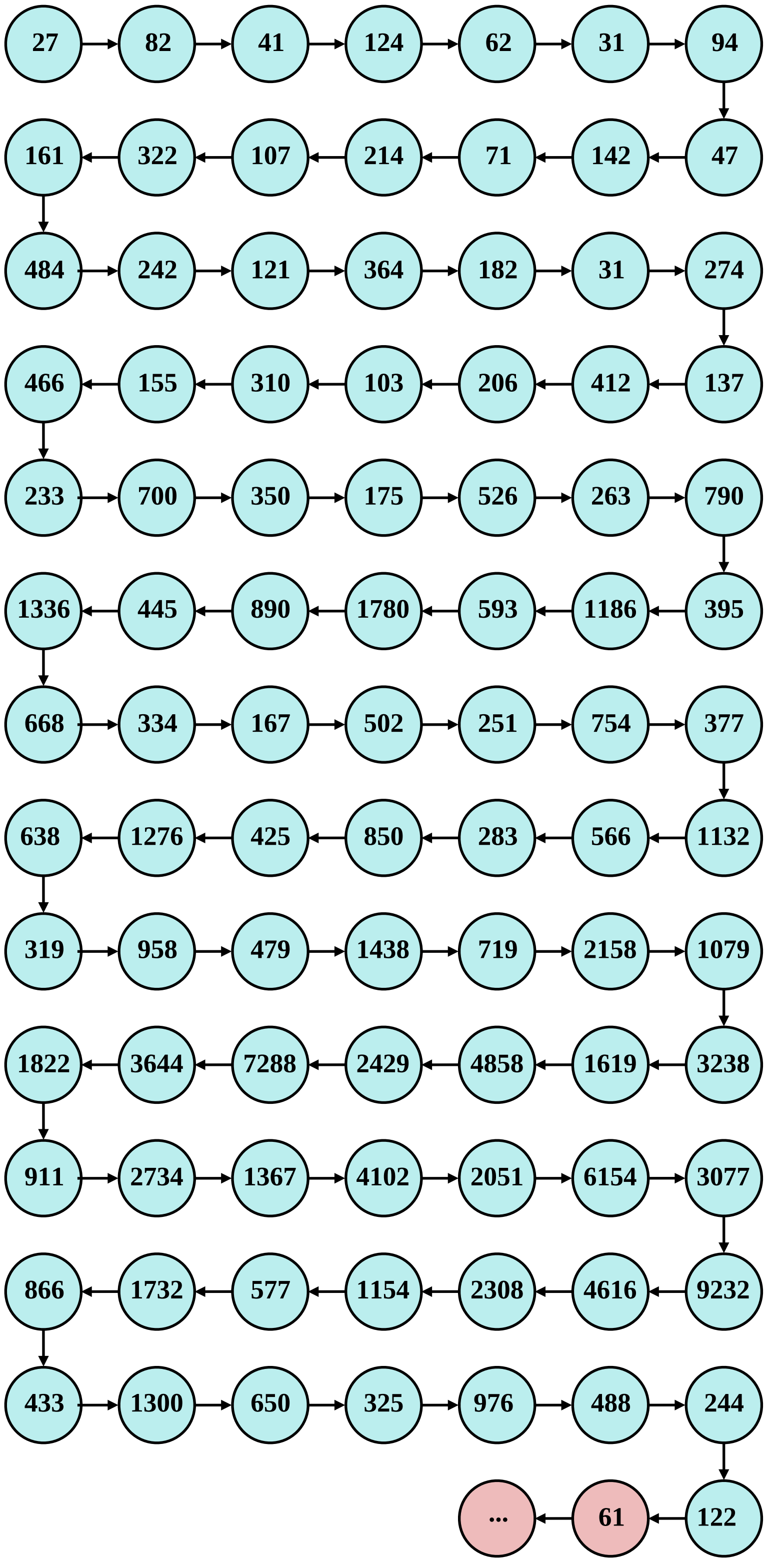
[12 marks]

(iv) Can you get four, five, six, ... consecutive starting values with the same *Total Stopping Time* ?

[4 marks]

3.3 Answers

3.3.1 Solutions to 3.1 The Monster 27 Sequence



[6 marks]

The updated Collatz 100 Table is;

Collatz 100 Table

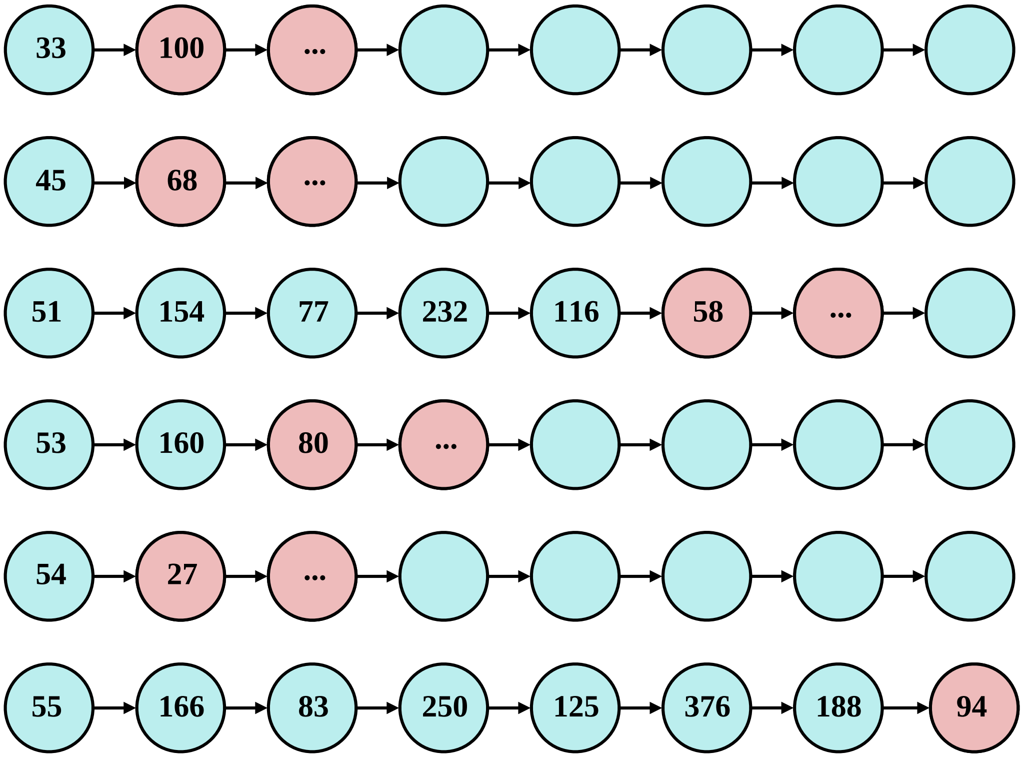
1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

[**2 marks]**

3.3.1 Solutions to 3.2 Exercise

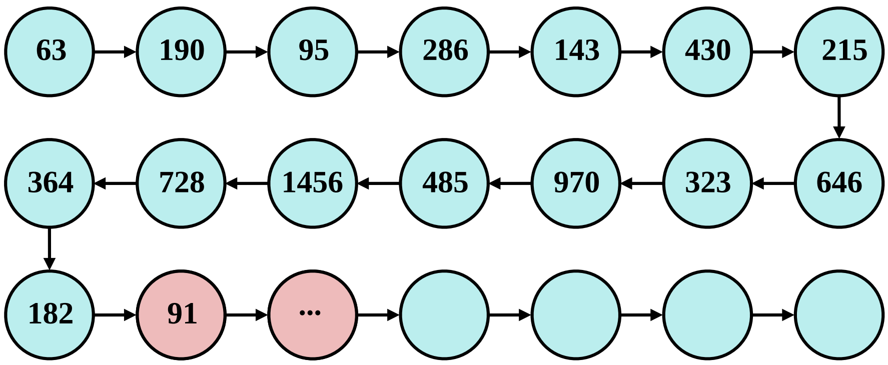
Answer 1

(i) Starting to working through the numbers not yet accounted for...



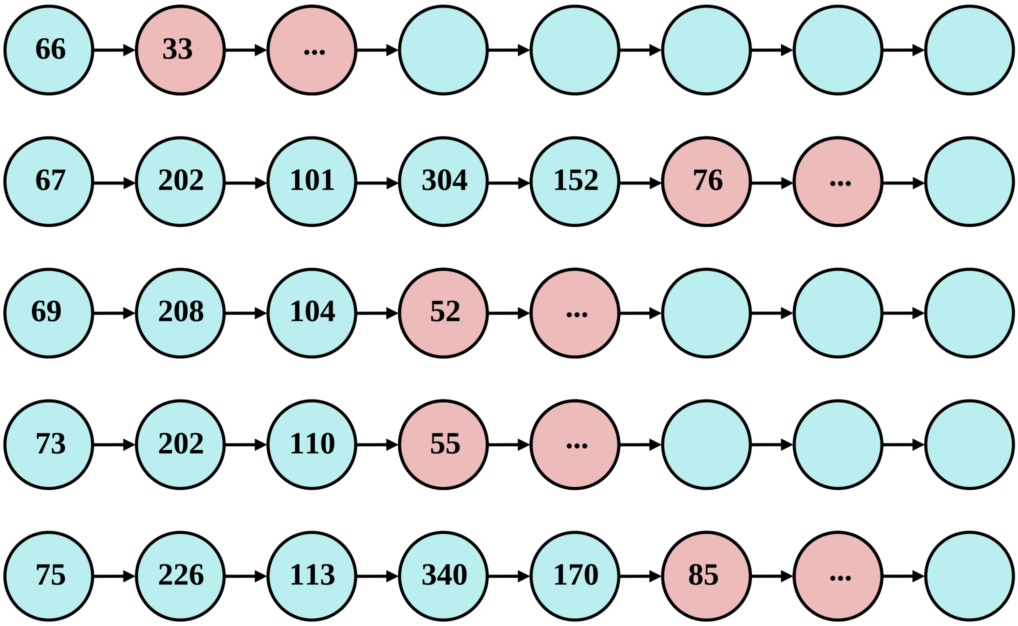
[6 marks]

(ii) A longer sequence...



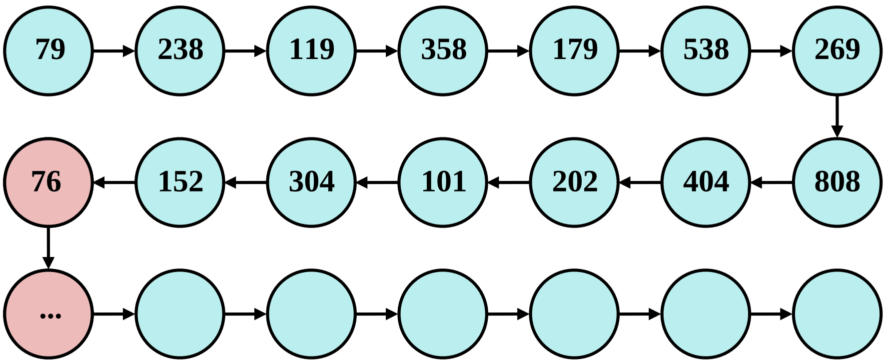
[3 marks]

(iii) Some more that join on to known numbers quickly...



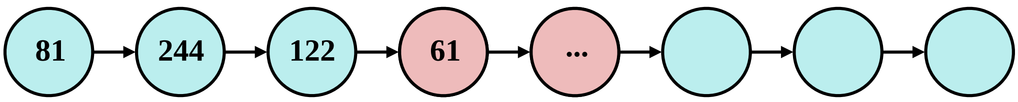
[5 marks]

(iv) Another longer sequence ...



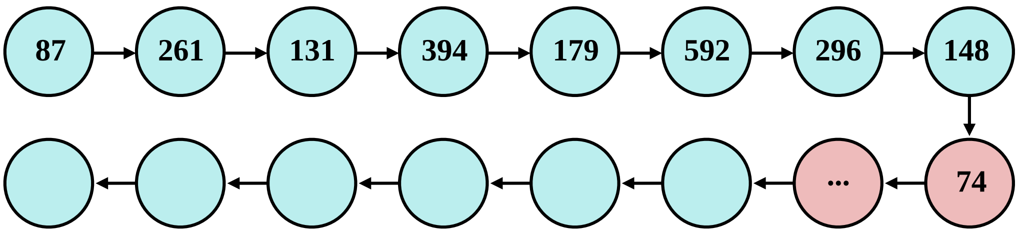
[3 marks]

(v) Short ...



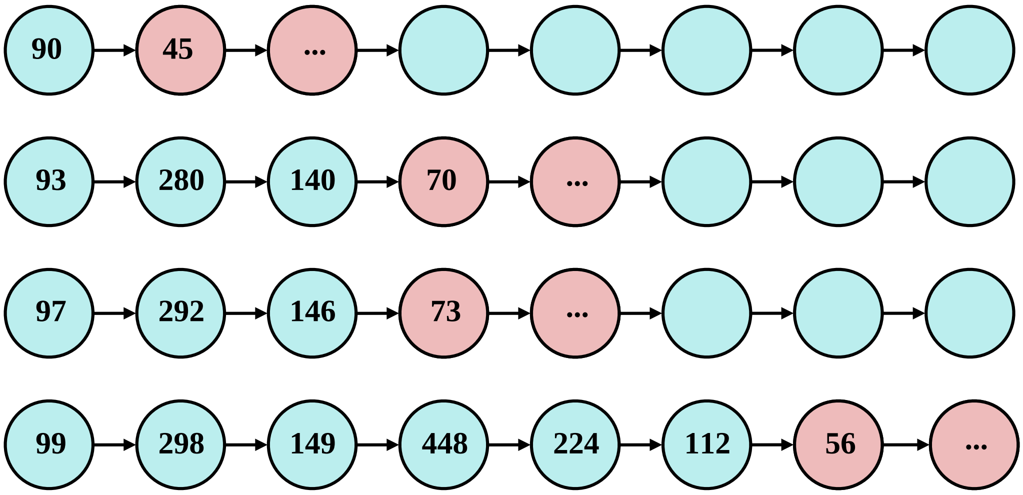
[1 mark]

(vi) Long ...



[2 marks]

(vii) Going for the finish ...



[4 marks]

Answer 2

(i) 118

[2 marks]

(ii) 97

[2 marks]

(iii)

Total Stopping Times 101 - 200

	1	2	3	4	5	6	7	8	9	10
100	25	25	87	12	38	12	100	113	113	113
110	69	20	12	33	33	20	20	33	33	20
120	95	20	46	108	108	108	46	7	121	28
130	28	28	28	28	41	15	90	15	41	15
140	15	103	103	23	116	116	116	23	23	15
150	15	23	36	23	85	36	36	36	54	10
160	98	23	23	111	111	111	67	10	49	10
170	124	31	31	31	80	18	31	31	31	18
180	18	93	93	18	44	18	44	106	106	106
190	44	13	119	119	119	26	26	26	119	26

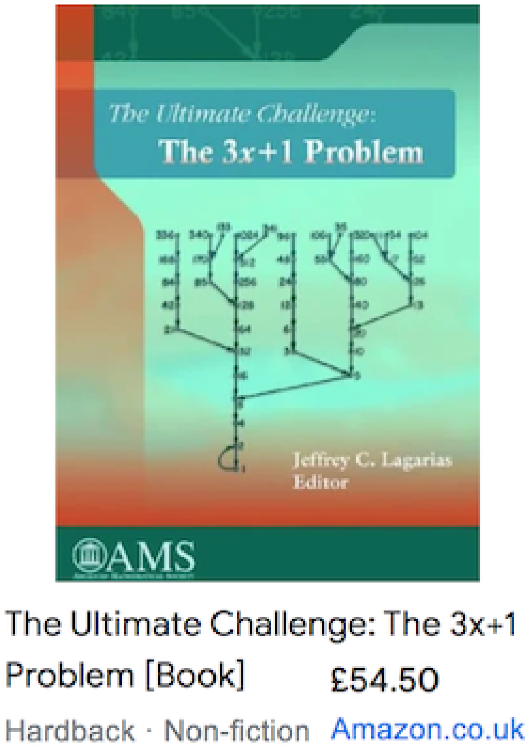
[12 marks]

(iv) Four: none found between 1 and 200
Five: five consecutive 28s are highlighted
(There's also five consecutive 25s on the join between the two tables)
Six: none found between 1 and 200

[4 marks]

4.1 The Sign Flipped Collatz Rule

Entire books have been written about the Collatz conjecture. The current best was published in 2010 and is titled *The Ultimate Challenge: The 3x+1 Problem.* It is written by Jeffrey Lagarias who is a mathematics professor at the University of Michigan, USA, and features a few guest chapters by other mathematicians who have spent time working on this problem.



In 2021 an updated version of the first chapter was made available free of charge on the arXiv website: https://arXiv.org/abs/2111.02635

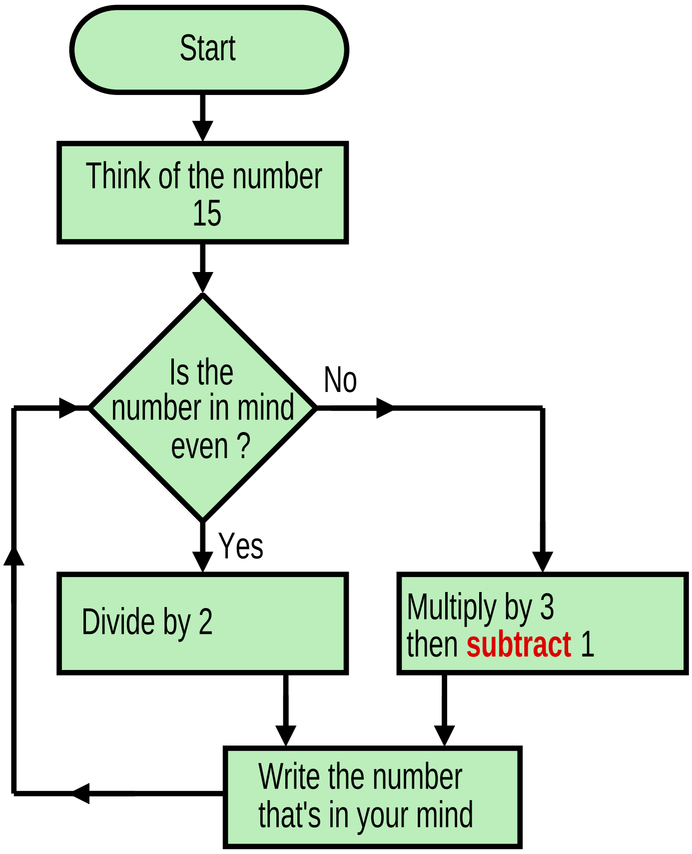
There is a good Numberphile video that summarises the Collatz Conjecture featuring Professor David Eisenbud (about 8 minutes).

Teaching Video : https://www.NumberWonder.co.uk/v9120/2.mp4

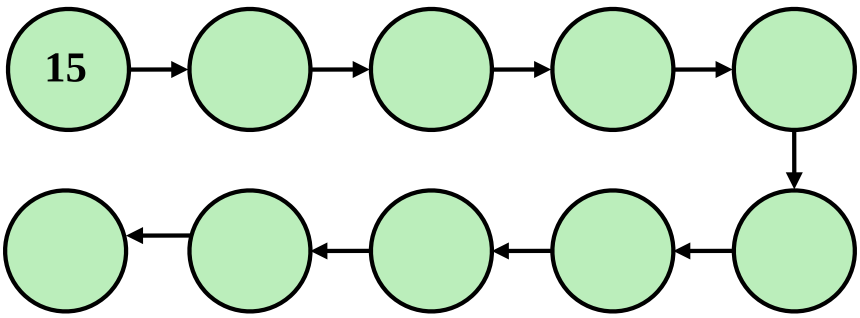
Towards the end of the video there is a suggestion that a similar iteration be looked at, *the sign flipped Collatz rule*, which is what our last lesson on the topic is about.

4.2 If At First You Don't Succeed

When a problem seems difficult, one avenue to explore is that of similar problems to see what sort of things happen there. Do they exhibit similar behaviours or do different things occur ?



On the diagram write the number sequence generated by the flowchart.



[4 marks]

This initial exploration of the *sign flipped Collatz rule*, given by,

$$y = \begin{cases} \dfrac{x}{2} & \text{if } x \text{ is even} \\ 3x - 1 & \text{if } x \text{ is odd} \end{cases}$$

seems to suggest that it's likely to behave in much the same way as the original rule.

However, prepare for some surprises as you work through the exercise !

4.3 Exercise

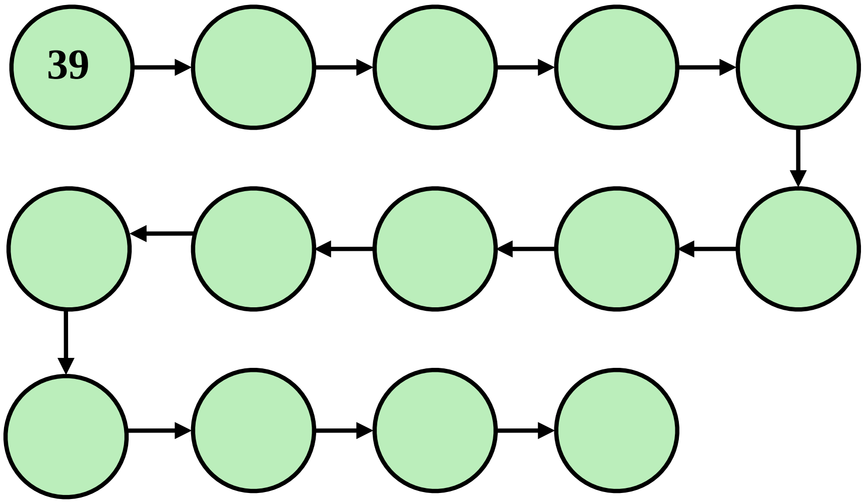
Marks Available : 38

Question 1

Use the *sign flipped Collatz rule*, given by,

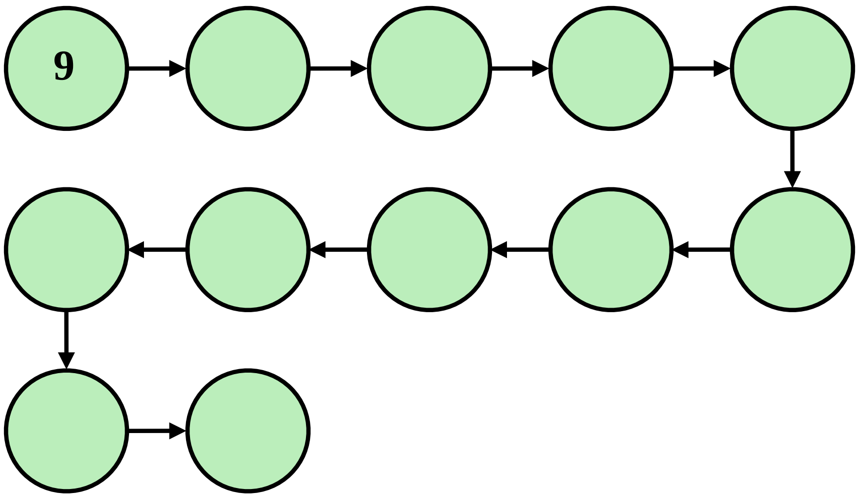
$$y = \begin{cases} \dfrac{x}{2} & \text{if } x \text{ is even} \\ 3x - 1 & \text{if } x \text{ is odd} \end{cases}$$

(i) with a starting number of 39.



[4 marks]

(ii) with a starting number of 9.



[4 marks]

(iii) Comment on your answers.

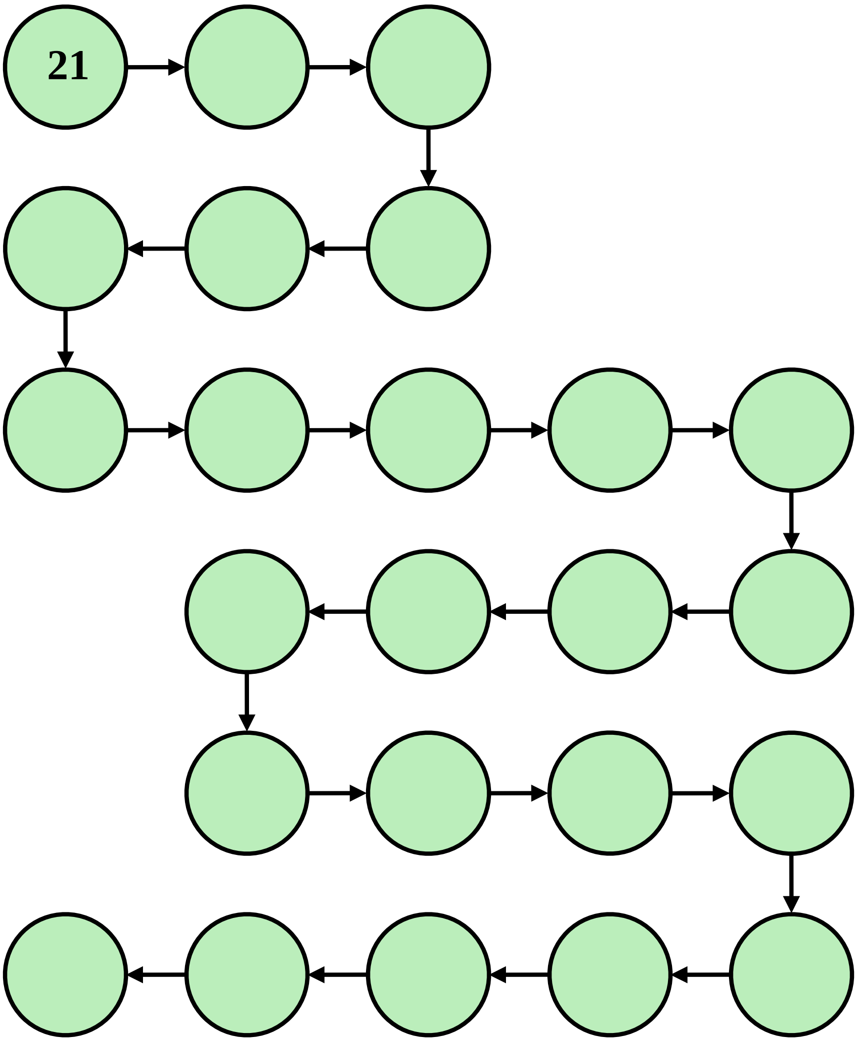
[2 marks]

Question 2

In this question we'll look a little more at the *sign flipped Collatz rule*,

$$y = \begin{cases} \dfrac{x}{2} & \text{if } x \text{ is even} \\ 3x - 1 & \text{if } x \text{ is odd} \end{cases}$$

(i) Complete the following diagram.



[6 marks]

(ii) Explain how the diagram shows the *sign flipped Collatz rule* can have different behaviour to the regular rule used by the Collatz conjecture.

[2 marks]

Question 3

Watts' Conjecture

Here is an iterative rule that's a variation on those previously looked at.

$$y = \begin{cases} \dfrac{x}{2} & \text{if } x \text{ is even} \\ 5x - 3 & \text{if } x \text{ is odd} \end{cases}$$

Watts' Conjecture claims that all positive integers will either

(a) Terminate in the 1-loop [1, 2]

or

(b) terminate in the 3-loop [3, 12, 6]

To get a good initial feel for this iterative rule, work through the sequence generated by starting from,

(i) The number 5

[4 marks]

(ii) The number 15

[4 marks]

(iii) The numbers less than 100 so far have partitioned as follows;

Watts' 100 Table

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

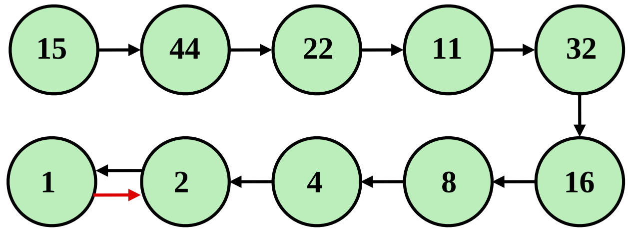
An interesting pattern seems to emerge if the start numbers that go to the 1-loop are coloured differently to those going to the 3-loop. Investigate if this pattern continues to hold.

[12 marks]

4.4 Answers

4.4.1 Solutions to 4.2 If At First You Don't Succeed

(i)

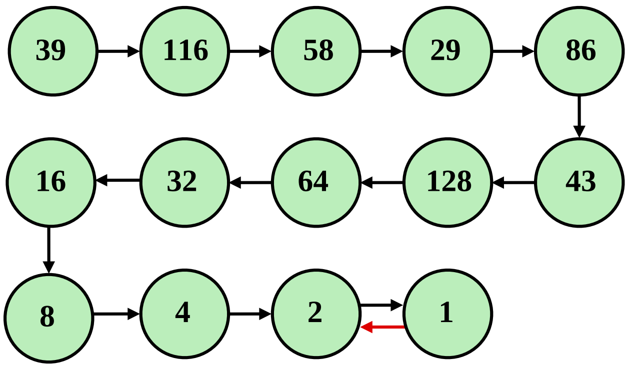


[4 marks]

4.4.2 Solution to 4.3 Exercise

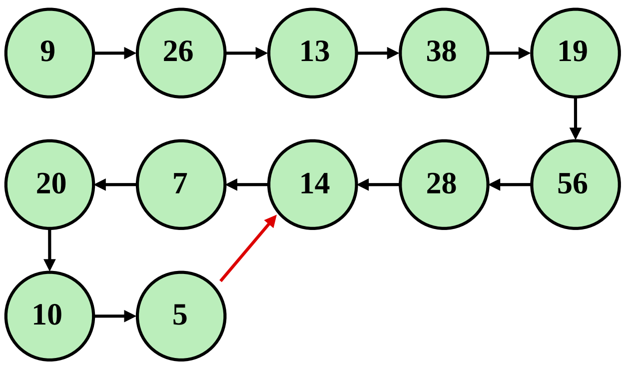
Answer 1

(i)



[4 marks]

(ii)



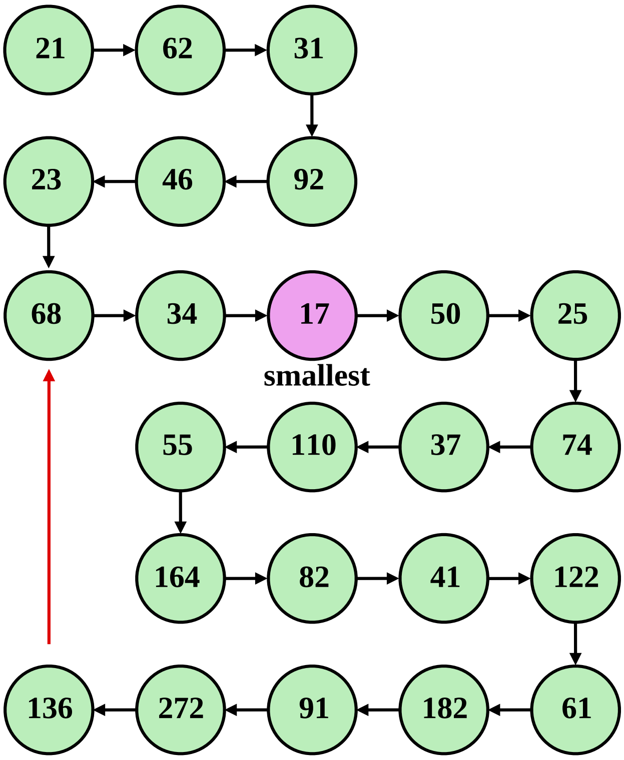
[4 marks]

(iii) Answer 1 exhibits behaviour like the regular Collatz sequence we've previously looked at falling into a loop with 1 as smallest number.
Call this the 1-loop.
However, answer 2 shows that a Collatz like conjecture for the sign flipped sequence would not be true as this sequence never gets to 1.
We've exhibited a number, 9, that does not “fall to earth”.
Instead it falls into a loop where 5 is the smallest number.
Call this the 5-loop.

[2 marks]

Answer 2

(i)



[6 marks]

(ii) The 21 falls into another, different, loop.
The smallest number in the loop is 17, so call it the 17-loop.
The equivalent of the Collatz conjecture is (again) NOT true for this iteration.
Numbers seem to either fall into the 1-loop, or 5-loop or 17-loop.
It's not currently known if there are any other loops for this iteration.

[2 marks]

Extra : If a sign flipped Collatz 100 table is made, and entries coloured according to which loop they end up in the following results.

Sign Flipped Collatz Rule 100 Table

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

There does not seem to be any obvious pattern...

Answer 3

(i) $5 \Rightarrow 22 \Rightarrow 11 \Rightarrow 52 \Rightarrow 26 \Rightarrow 13 \Rightarrow 62 \Rightarrow 31 \Rightarrow 152 \Rightarrow 76 \Rightarrow 38$
$\Rightarrow 19 \Rightarrow 92 \Rightarrow 46 \Rightarrow 23 \Rightarrow 112 \Rightarrow 56 \Rightarrow 28 \Rightarrow 14 \Rightarrow 7 \Rightarrow 32 \Rightarrow$
$16 \Rightarrow 8 \Rightarrow 4 \Rightarrow$ [2, 1 loop]

[4 marks]

(ii) $15 \Rightarrow 72 \Rightarrow 36 \Rightarrow 18 \Rightarrow 9 \Rightarrow 42 \Rightarrow 21 \Rightarrow 102 \Rightarrow 51 \Rightarrow 252$
$\Rightarrow 126 \Rightarrow 63 \Rightarrow 312 \Rightarrow 156 \Rightarrow 78 \Rightarrow 39 \Rightarrow 192 \Rightarrow 96 \Rightarrow$
$48 \Rightarrow 24 \Rightarrow$ [12, 6, 3 loop]

[4 marks]

(iii) An initial investigation suggests that the pattern may continue to hold. However, some numbers give sequences that include some numerically large integers. Sequences were terminated if a term greater than 2,147,483,647 occurred.
So, there is more work to do on this !

Watts' 100 Table

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

[12 marks]

Extra :

Pick an iterative rule that is similar to that of the Collatz conjecture and explore it, initially on paper.
Your rule should be of the form,

$$y = \begin{cases} \dfrac{x}{2} & \text{if } x \text{ is even} \\ a\,x \pm b & \text{if } x \text{ is odd} \end{cases}$$

for some integer values of a and b that you have chosen.
You also get to pick if the sign is + or –

ITERATION

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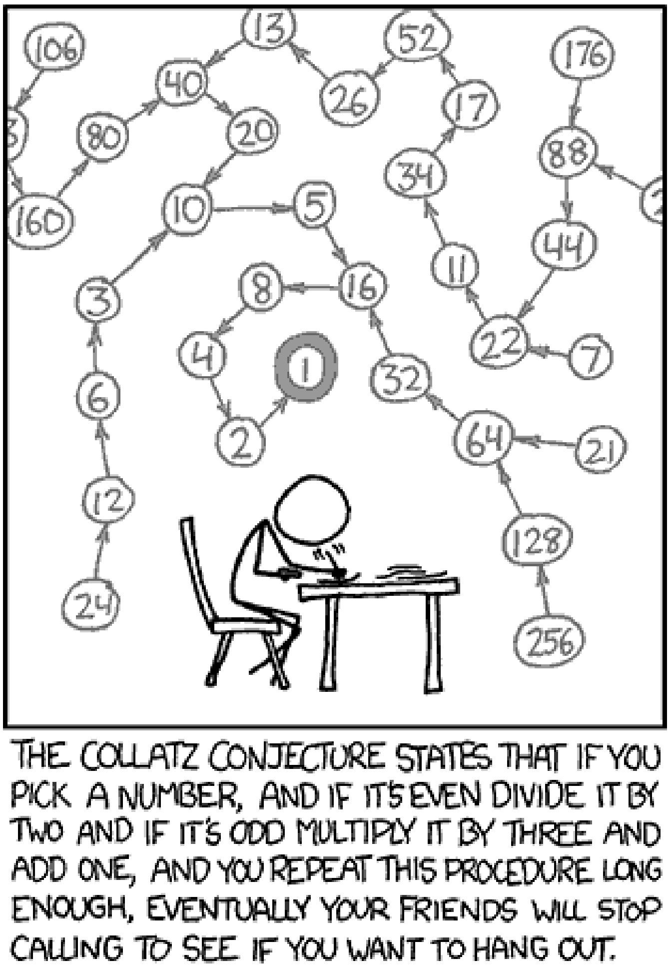

Detailed worked solutions to the exercises are available from m.hamsen2@unimail.derby.ac.uk