The mean value of a function

Learning Objectives

- To be able to find the mean value of a function and apply transformations.
- To be able to apply a range of integration techniques including by substitution and by parts.

Starter Questions

1. $\int 2x^2(x^3-1)^3\,dx$

Substitution

$$\boxed{\begin{aligned} &\text{Let } u = x^3-1 \\ &\frac{du}{dx} = 3x^2 \\ &\frac{du}{3x^2} = dx \end{aligned}}$$

$$\int 2x^2 . u^3 . \frac{du}{3x^2}$$

$$\frac{2}{3}\int u^3 . du$$

$$\frac{2}{3} . \frac{u^4}{4} + c = \underline{\frac{1}{6}(x^3-1) + c}$$

2. $\int \frac{\sin\theta}{1+\cos 2\theta}\,d\theta$

Use $\cos 2\theta \equiv 2\cos^2\theta - 1$

$$\int \frac{\sin\theta}{1+2\cos^2\theta-1}\,d\theta$$

$$\frac{1}{2}\int \frac{\sin\theta}{\cos^2\theta}\,d\theta$$

Substitution

$$\boxed{\begin{aligned} &\text{Let } u = \cos\theta \\ &\frac{du}{d\theta} = -\sin\theta \\ &\frac{du}{-\sin\theta} = d\theta \end{aligned}}$$

$$\frac{1}{2}\int \frac{\sin\theta}{u^2} . \frac{du}{-\sin\theta}$$

$$\frac{-1}{2}\int u^{-2} . du$$

$$= \frac{1}{2}u^{-1} + c$$

$$= \underline{\frac{1}{2\cos\theta} + c}$$

$$\int u \, dv = uv - \int v \, du$$

3. $\int_0^{\frac{\pi}{6}} x \sin(3x)\, dx$

Let $u = x$ $\quad dv = \sin(3x)\, dx$

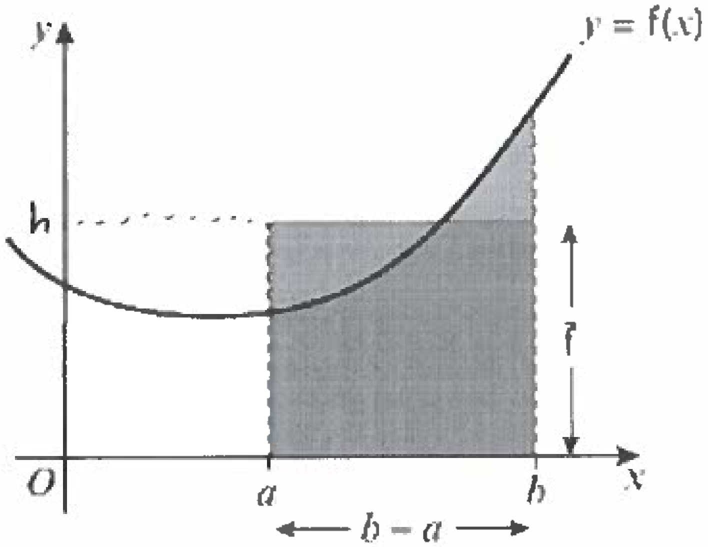
$\frac{du}{dx} = 1$ $\quad v = -\frac{1}{3}\cos 3x$

$$\int_0^{\pi/6} x \sin 3x \, dx = x \times \frac{-1}{3}\cos 3x + \int \frac{1}{3} \cdot \cos 3x \, dx$$

$$= \left[-\frac{x}{3}\cos 3x + \frac{1}{9}\sin 3x \right]_0^{\pi/6} = \left(0 + \frac{1}{18}\right) - (0 + 0)$$

$$= 1/18$$

Mean Value Function



Area under the curve $= \int_a^b f(x) . dx$

At a y-value of h, the area of the rectangle is equal to the area under the curve then:

$$(b-a)h = \int_a^b f(x). dx$$

$$h = \frac{1}{b-a} \int_a^b f(x). dx$$

$\therefore$ h is the mean value of a function $\bar{f} = h$.

Example 1

Find the mean value of $f(x) = \dfrac{4}{\sqrt{2 + 3x}}$ over the interval [2, 6].

"reverse chain rule"

$$\int 4(2+3x)^{-1/2}\,.dx = \frac{8}{3}(2+3x)^{1/2} + c$$

$$\bar{f} = \frac{1}{6-2}\int_2^6 4(2+3x)^{-1/2}\,dx$$

$$\bar{f} = \frac{1}{4} \times \frac{8}{3}\left[\sqrt{2+3x}\right]_2^6 = \frac{2}{3}\left(\sqrt{20} - \sqrt{8}\right)$$

$$= \frac{4}{3}\left(\sqrt{5} - \sqrt{2}\right)$$

Example 2

Find the exact mean value of
$f(x) = x\cos 2x$ over the interval $\left[0, \frac{\pi}{2}\right]$.

$$\int x\cos 2x\,.dx$$

Let $u = x$ $\quad dv = \cos 2x\,.dx$

$\frac{du}{dx} = 1$ $\quad v = \frac{1}{2}\sin 2x$

$$\int x\cos 2x\,dx = \frac{1}{2}x\sin 2x - \int \frac{1}{2}\sin(2x)\,.dx$$

$$= \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + c$$

$$\bar{f} = \frac{1}{\frac{\pi}{2}-0}\int_0^{\pi/2} x\cos 2x\,.dx = \frac{2}{\pi}\left[\frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x\right]_0^{\pi/2}$$

$$\bar{f} = \frac{2}{\pi}\left[\left(0\right) - \frac{1}{4}\right) - \left(0 + \frac{1}{4}\right) = \frac{2}{\pi} \times \frac{-1}{2} = \frac{-1}{\pi}$$

Question Practice

1. Find the exact mean value of $f(x) = \dfrac{\sin x \cos x}{\cos 2x + 2}$ over the interval $\left[0, \dfrac{\pi}{2}\right]$.

$$\int \frac{\sin x \cos x}{\cos 2x + 2}\,dx = \frac{1}{2}\int \frac{\sin 2x}{\cos 2x + 2}\,dx$$

Let $u = \cos 2x$

$\dfrac{du}{dx} = -2\sin 2x$

$$= \frac{1}{2}\int \frac{\sin 2x}{u+2} \cdot \frac{du}{-2\sin 2x}$$

$$= \frac{-1}{4}\int \frac{1}{u+2}\,du = \frac{-1}{4}\ln(u+2) + c$$

$$\bar{f} = \frac{1}{\frac{\pi}{2} - 0}\int_0^{\pi/2} \frac{\sin x \cos x}{\cos 2x + 2}\,dx = \frac{2}{\pi} \times \frac{-1}{4}\Big[\ln(\cos 2x + 2)\Big]_0^{\pi/2}$$

$$\bar{f} = \frac{-1}{2\pi}\left(0 - \ln(3)\right) \rightarrow \bar{f} = \frac{\ln(3)}{2\pi}$$

2. Find the exact mean value of $f(x) = x \sin 2x$ over the interval $\left[0, \dfrac{\pi}{3}\right]$.

$$\int x \sin 2x\,dx = \frac{-x}{2}\cos 2x + \frac{1}{2}\int \cos 2x\,dx$$

$u = x \qquad dv = \sin 2x\,dx$

$\dfrac{du}{dx} = 1 \qquad v = \dfrac{-1}{2}\cos 2x$

$$= \frac{-x}{2}\cos 2x + \frac{1}{4}\sin 2x + c$$

$$\bar{f} = \frac{1}{\frac{\pi}{3} - 0}\int_0^{\pi/3} x \sin 2x\,dx = \frac{3}{\pi}\left[\frac{-x}{2}\cos(2x) + \frac{1}{4}\sin(2x)\right]_0^{\pi/3}$$

$$\bar{f} = \frac{3}{\pi}\left[\left(\left(\frac{-\pi}{6} \times \frac{-1}{2}\right) + \frac{\sqrt{3}}{8}\right) - (0 + 0)\right]$$

$$\bar{f} = \frac{1}{4} + \frac{3\sqrt{3}}{8\pi}$$

If the function f(x) has mean value $\bar{f}$ over the interval [a, b], and k is a real constant, then:

- **f(x) + k has mean value $\bar{f} + k$ over the interval [a, b]**
- **kf(x) has mean value $k\bar{f}$ over the interval [a, b]**
- **–f(x) has mean value –$\bar{f}$ over the interval [a, b].**

Watch out You cannot deduce the mean value of f(–x) or f(kx) in this way.

3.

$f(x) = x(x^2 - 4)^4$

a Show that the mean value of f(x) over the interval [0, 2] is $\frac{256}{5}$ **(3 marks)**

b Use the answer to part **a** to find the mean value over the interval [0, 2] of $-2f(x)$. **(2 marks)**

a) $\int x(x^2-4)^4\,dx$

Let $u = x^2 - 4$

$\frac{du}{dx} = 2x$

$\int x(u^4)\frac{du}{2x}$

$\frac{1}{2}\int u^4\,du = \frac{1}{10}u^5 + c = \frac{1}{10}(x^2-4)^5 + c$

$\bar{f} = \frac{1}{2-0}\int_0^2 x(x^2-4)^4\,dx$

$\bar{f} = \frac{1}{20}\left[(x^2-4)^5\right]_0^2 = \frac{1}{20}\left((0) - (-1024)\right)$

$\bar{f} = 256/5$

b) $-2 \times \frac{256}{5} = \frac{-512}{5}$

4. $f(x) = \ln(kx)$, where k is a positive constant.

Given that the mean value of $f(x)$ on the interval $[0, 2]$ is -2, find the value of k. **(4 marks)**

$$\int \ln(kx)\,dx \qquad u = \ln(kx) \qquad dv = dx$$

$$= x\ln(kx) - \int 1 \,.\, dx \qquad \frac{du}{dx} = \frac{1}{x} \qquad v = x$$

$$= x\ln(kx) - x + c$$

$$\bar{f} = \frac{1}{2-0}\int_0^2 \ln(kx)\,.\,dx = -2$$

$$\frac{1}{2}\left[x\ln(kx) - x\right]_0^2 = -2$$

$$\ln(2k) - 1 = -2 \rightarrow \ln(2k) = -1$$

$$2k = e^{-1}$$

$$k = \frac{1}{2}e^{-1}$$

5. $f(x) = \dfrac{\cos x}{(2 + \sin x)^2}$

a Find $\int f(x)\,dx$. **(4 marks)**

b Hence show that the mean value of $f(x)$ over the interval $\left[0, \frac{5\pi}{3}\right]$ is $-\frac{3}{130\pi}(3 + 4\sqrt{3})$. **(2 marks)**

a)

$$\int \frac{\cos x}{u^2} \cdot \frac{du}{\cos x} \qquad \text{Let } u = 2 + \sin x \qquad \frac{du}{dx} = \cos x$$

$$\int \frac{1}{u^2} \,.\, du = \frac{-1}{u} + c = \frac{-1}{2 + \sin x} + c$$

b)

$$\bar{f} = \frac{1}{\frac{5\pi}{3} - 0}\int \frac{\cos x}{(2+\sin x)^2}\,.\,dx$$

$$\bar{f} = \frac{3}{5\pi}\left[\frac{-1}{(2+\sin x)}\right]_0^{5\pi/3}$$

$$\bar{f} = \frac{3}{5\pi}\left[\left(\frac{-1}{2 - \sqrt{3}/2}\right) + 1/2\right] \rightarrow \bar{f} = \frac{3}{5\pi}\left(\frac{2}{\sqrt{3}-4} + \frac{1}{2}\right)$$

$$\bar{f} = \frac{3}{5\pi}\left(\frac{2\sqrt{3}+8}{-13} + \frac{1}{2}\right)$$

$$\bar{f} = \frac{-3}{130\pi}\left(-4\sqrt{3} + 3\right).$$

Exam Question – June 2019 Core Pure Paper 2

3.

$$f(x) = \frac{1}{\sqrt{4x^2+9}}$$

(a) Using a substitution, that should be stated clearly, show that

$$\int f(x)\,dx = A\sinh^{-1}(Bx) + c$$

where c is an arbitrary constant and A and B are constants to be found.

(4)

(b) Hence find, in exact form in terms of natural logarithms, the mean value of f(x) over the interval [0, 3].

(2)

a) $\displaystyle\int \frac{1}{\sqrt{4\left(x^2+9/4\right)}}\,dx$

$\displaystyle\int \frac{1}{\sqrt{4}\sqrt{x^2+9/4}}\,dx$

$\displaystyle\frac{1}{2}\int \frac{1}{\sqrt{x^2+9/4}}\,dx$

$\frac{1}{2}\,\mathrm{arsinh}\left(\frac{2}{3}x\right)+c$ use.

$A=\frac{1}{2}\quad B=\frac{2}{3}$

Other option is allowed.

Using substitution

$x=\frac{3}{2}\sinh u$

$dx=\frac{3}{2}\cosh u\,du$

$\displaystyle\frac{1}{3}\int\frac{1}{\sqrt{\sinh^2 u+1}}\cdot\frac{3}{2}\cosh u\,du$

$\displaystyle\frac{1}{2}\int 1.\,du = \frac{1}{2}u+c$

$u=\mathrm{arsinh}\left(\frac{2}{3}x\right)$

NB: From the formula booklet

Integration (+ constant; $a > 0$ where relevant)

f(x)	$\int f(x)\,dx$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\ln\cosh x$
$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin\left(\frac{x}{a}\right)\quad(\lvert x\rvert < a)$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\arctan\left(\frac{x}{a}\right)$
$\frac{1}{\sqrt{x^2-a^2}}$	$\mathrm{arcosh}\left(\frac{x}{a}\right),\ \ln\{x+\sqrt{x^2-a^2}\}\quad(x > a)$
$\frac{1}{\sqrt{a^2+x^2}}$	$\mathrm{arsinh}\left(\frac{x}{a}\right),\ \ln\{x+\sqrt{x^2+a^2}\}$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert=\frac{1}{a}\mathrm{artanh}\left(\frac{x}{a}\right)\quad(\lvert x\rvert < a)$
$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert$

use

b) $\bar{f}=\frac{1}{3}\left[\frac{1}{2}\,\mathrm{arsinh}\left(\frac{2}{3}x\right)\right]_0^3$

$\bar{f}=\frac{1}{6}\left[\ln\left(x+\sqrt{x^2+\frac{9}{4}}\right)\right]_0^3$

$\bar{f}=\frac{1}{6}\left(\ln\left(3+\frac{3\sqrt{5}}{2}\right)\right)-\frac{1}{6}\ln\left(3/2\right)=\frac{1}{6}\ln\left(\frac{\frac{6+3\sqrt{5}}{2}}{3/2}\right)=\frac{1}{6}\ln\left(2+\sqrt{5}\right)$