1.3 Answers

1.3.1 Solution to 1.2 Exercise

Answer 1

(i) Yes, because $\frac{d^2x}{dt^2} = -9x$ is of the form $\frac{d^2x}{dt^2} = -\omega^2 x$ with $\omega = 3$

which is the equation that characterises simple harmonic motion.

[1 mark]

(ii)

$$\frac{d^2x}{dt^2} + 9x = 0$$

$$m^2 + 9 = 0 \quad \text{is the auxiliary equation}$$

$$m^2 - (3\mathrm{i})^2 = 0$$

$$m^2 = (3\mathrm{i})^2$$

$$m = \pm 3\mathrm{i} \qquad \text{(complex roots)}$$

The general solution is $x = e^{0t}(A\cos 3t + B\sin 3t)$

$$x = A\cos 3t + B\sin 3t$$

$$\frac{dx}{dt} = -3A\sin 3t + 3B\cos 3t$$

Using the boundary conditions, at $t = 0$, $x = 1$ and the velocity is 4 m.s^{-1}

$x = A\cos 3t + B\sin 3t \qquad \Rightarrow 1 = A$

$\frac{dx}{dt} = -3A\sin 3t + 3B\cos 3t \qquad \Rightarrow 4 = 3B$ so $B = \frac{4}{3}$

The particular solution is, $x = \cos 3t + \frac{4}{3}\sin 3t$

[5 marks]

(iii) In general, $x = R\sin(\omega t + \alpha) = R\sin\omega t\cos\alpha + R\cos\omega t\sin\alpha$

so $\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{1 \times 3}{4}$

$$\alpha = 0.644 \qquad \text{and} \qquad R = \sqrt{1^2 + \left(\frac{4}{3}\right)^2} = \frac{5}{3}$$

$x = \frac{5}{3}\sin(3t + 0.644)$ giving $x_{MAX} = \frac{5}{3}$ metres

First occurs when $\sin(3t + 0.644) = 1$, when, $3t + 0.644 = \frac{\pi}{2}$

Solve this to get $t = 0.309$ seconds.

[4 marks]

Answer 2

(i) $a = \frac{dv}{dt}$ and so $a = t + 5v$

becomes $\frac{dv}{dt} = t + 5v$

$\therefore \frac{dv}{dt} - 5v = t \quad \square$

[1 mark]

(ii) $a = \frac{d^2x}{dt^2}$ and $v = \frac{dx}{dt}$ so $a = t + 5v$

can be recast as $\frac{d^2x}{dt^2} = t + 5\frac{dx}{dt}$

$$\frac{d^2x}{dt^2} - 5\frac{dx}{dt} = t$$

This is not in the form of the differential equation for SHM

[2 marks]

(iii) $\int P(x)\ dt = \int (-5)\ dt = -5t$ so integrating factor is e^{-5t}

[1 mark]

(iv) $$\frac{dv}{dt} - 5v = t$$

Multiply through by the integrating factor

$$e^{-5t}\frac{dv}{dt} - 5e^{-5t}v = t\,e^{-5t}$$

integrate both sides with respect to t

$$e^{-5t}v = -\frac{1}{5}\int t\ (-5)\ e^{-5t}\,dt$$

$$= -\frac{1}{5}\left(t\,e^{-5t} - \int e^{-5t}\,dt + c_1\right)$$

$$= -\frac{1}{25}\left(5t\,e^{-5t} + \int(-5)\ e^{-5t}\,dt + c_1\right)$$

$$= -\frac{1}{25}\left(5t\,e^{-5t} + e^{-5t} + c_2\right) +$$

$$v = \frac{1}{25}\left(-5t - 1 + c_3\,e^{5t}\right)$$

Use the boundary condition $v = 0$ when $t = 0$: $0 = \frac{1}{25}\left(0 - 1 + c_3\right)$

$$c_3 = 1$$

$$\therefore v = \frac{1}{25}\left(e^{5t} - 5t - 1\right)$$

[5 marks]

Answer 3

(i) It's simple harmonic motion.

[1 mark]

(ii) The equation is of the form $\dfrac{d^2x}{dt^2} = -\omega^2 x$

Using the information given, $-\ 5 = -\omega^2 \times 1$ so $\omega^2 = 5$

$$\therefore\ \frac{d^2x}{dt^2} = -5x$$

[2 marks]

(iii)

$$\frac{d^2x}{dt^2} + 5x = 0$$

$m^2 + 5 = 0$ is the auxiliary equation

$$m^2 - \left(\sqrt{5}\ \mathrm{i}\right)^2 = 0$$

$$m = \pm\sqrt{5}\ \mathrm{i} \quad \text{(complex roots)}$$

$$x = A\,cos\sqrt{5}\ t + B\,sin\sqrt{5}\ t$$

$$\frac{dx}{dt} = -\sqrt{5}\ A\,sin\sqrt{5}\ t + \sqrt{5}\ B\,cos\sqrt{5}\ t$$

$$\frac{d^2x}{dt^2} = -5A\,cos\sqrt{5}\ t - 5B\,sin\sqrt{5}\ t$$

Using the boundary conditions that at $t = 0$, $v = 6$ and $x = 5$

$x = A\,cos\sqrt{5}\ t + B\,sin\sqrt{5}\ t \Rightarrow 5 = A$

$\dfrac{dx}{dt} = -\sqrt{5}\ A\,sin\sqrt{5}\,t + \sqrt{5}\ B\,cos\sqrt{5}\ t \Rightarrow 6 = \sqrt{5}\ B$ so $B = \dfrac{6\sqrt{5}}{5}$

$$\therefore\ x = 5\,cos\sqrt{5}\ t + \frac{6\sqrt{5}}{5}\ sin\sqrt{5}\ t$$

[7 marks]

(iv)

$$D_{MAX} = \sqrt{5^2 + \left(\frac{6\sqrt{5}}{5}\right)^2}$$

$$= \frac{\sqrt{805}}{5}$$

[2 marks]

Answer 4

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = \cos 3t$$

(a) Obtain the complementary function by solving the homogeneous equation,

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = 0$$

$$m^2 + 6m + 9 = 0 \text{ is the auxiliary equation}$$

$$(m + 3)^2 = 0$$

$$m = -3 \qquad \text{(repeated root)}$$

$\therefore$ The complementary function is $x = (A + Bx)\, e^{-3t}$

For a particular integral try $x = U\cos 3t + V\sin 3t$

in which case $\dfrac{dx}{dt} = -3U\sin 3t + 3V\cos 3t$

and $\dfrac{d^2x}{dt^2} = -9U\cos 3t - 9V\sin 3t$

Substitute these into the question's differential equation...

$$-9U\cos 3t - 9V\sin 3t - 18U\sin 3t + 18V\cos 3t + 9U\cos 3t + 9V\sin 3t = \cos 3t$$

$$(-9U + 18V + 9U)\cos 3t + (-9V - 18U + 9V)\sin 3t = \cos 3t$$

$$18V = 1 \text{ so } V = \frac{1}{18}$$

$$-18U = 0 \text{ so } U = 0$$

$\therefore$ The particular integral is $x = \dfrac{1}{18}\sin 3t$

The general solution is $x = (A + Bt)\, e^{-3t} + \dfrac{1}{18}\sin 3t$

[8 marks]

(b) $\dfrac{dx}{dt} = (A + Bt)(-3)e^{-3t} + Be^{-3t} + \dfrac{1}{6}\cos 3t$

Use the boundary conditions: when $t = 0$, $x = \dfrac{1}{2}$ and $\dfrac{dx}{dt} = 0$

$x = (A + Bt)\, e^{-3t} + \dfrac{1}{18}\sin 3t \Rightarrow A = \dfrac{1}{2}$

$\dfrac{dx}{dt} = (A + Bt)(-3)e^{-3t} + Be^{-3t} + \dfrac{1}{6}\cos 3t$

$$\Rightarrow -\frac{3}{2} + B + \frac{1}{6} = 0 \text{ so } B = \frac{4}{3}$$

The particular solution is $x = \left(\dfrac{1}{2} + \dfrac{4}{3}t\right) e^{-3t} + \dfrac{1}{18}\sin 3t$

[5 marks]

(c) From $\dfrac{dx}{dt} = (A + Bt)(-3)e^{-3t} + Be^{-3t} + \dfrac{1}{6}\cos 3t$

with the values of A and B just worked out,

$$\frac{dx}{dt} = \left(\frac{1}{2} + \frac{4}{3}t\right)(-3)e^{-3t} + \frac{4}{3}e^{-3t} + \frac{1}{6}\cos 3t$$

For a turning point, we require,

$$\left(\frac{1}{2} + \frac{4}{3}t\right)(-3)e^{-3t} + \frac{4}{3}e^{-3t} + \frac{1}{6}\cos 3t = 0$$

For $t > 30$, the terms involving e^{-3t} will be negligible.

$\therefore$ Solve $\cos 3t = 0$

$$3t = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots, \frac{(2n-1)\pi}{2}, \ldots$$

$$t = \frac{\pi}{6}, \frac{3\pi}{6}, \ldots, \frac{(2n-1)\pi}{6}, \ldots$$

We want $\dfrac{(2n-1)\pi}{6} > 30$

$$2n - 1 > \frac{30 \times 6}{\pi}$$

$$2n - 1 > 57.3$$

$$n > 29.1 \Rightarrow n = 30 \text{ (must be an integer)}$$

$$t = \frac{(2 \times 30 - 1)\pi}{6}$$

$$= \frac{59\pi}{6}$$

$$x = \frac{1}{18}\sin\left(\frac{3 \times 59\pi}{6}\right)$$

$$= -\frac{1}{18}$$

The coordinates of A are thus $\left(\dfrac{59\pi}{6}, -\dfrac{1}{18}\right)$

[2 marks]

Answer 5

(a) **(i)**

$$\theta = \frac{1}{12}\, t \sin 3t$$

$$\frac{d\theta}{dt} = \frac{1}{4}\, t \cos 3t + \frac{1}{12} \sin 3t$$

$$\frac{d^2\theta}{dt^2} = -\frac{3}{4}\, t \sin 3t + \frac{1}{4} \cos 3t + \frac{1}{4} \cos 3t$$

$$= \frac{1}{2} \cos 3t - \frac{3}{4}\, t \sin 3t$$

Now to show that these bits satisfy the given equation...

$$\text{LHS} = \frac{d^2\theta}{dt^2} + 9\theta$$

$$= \frac{1}{2} \cos 3t - \frac{3}{4}\, t \sin 3t + 9\left(\frac{1}{12}\, t \sin 3t\right)$$

$$= \frac{1}{2} \cos 3t - \frac{3}{4}\, t \sin 3t + \frac{3}{4}\, t \sin 3t$$

$$= \frac{1}{2} \cos 3t$$

$$= \text{RHS} \qquad \square$$

So the particular integral is indeed $\theta = \frac{1}{12}\, t \sin 3t$

[4 marks]

(ii) Obtaining the complementary function...

$$\frac{d^2\theta}{dt^2} + 9\theta = 0 \quad \text{homogeneous equation}$$

$$m^2 + 9 = 0 \quad \text{is the auxiliary equation}$$

$$m^2 - (3\mathrm{i})^2 = 0$$

$$m = \pm 3\mathrm{i} \quad \text{(complex roots)}$$

$\therefore$ The complementary function is $\theta = A \cos 3t + B \sin 3t$

General solution: $\theta = A \cos 3t + B \sin 3t + \frac{1}{12}\, t \sin 3t$

[4 marks]

(b) Now to use the boundary conditions, when $t = 0$, $\theta = \dfrac{\pi}{3}$ and $\dfrac{d\theta}{dt} = 0$

$$\theta \;=\; A\,cos\,3t \;+\; B\,sin\,3t \;+\; \frac{1}{12}\,t\,sin\,3t \;\Rightarrow\; A \;=\; \frac{\pi}{3}$$

$$\frac{d\theta}{dt} \;=\; -\,3A\,sin\,3t \;+\; 3B\,cos\,3t \;+\; \frac{3}{12}\,t\,cos\,3t \;-\; \frac{1}{12}\,sin\,3t \;\Rightarrow\; B \;=\; 0$$

Particular solution: $\theta \;=\; \dfrac{\pi}{3}\,cos\,3t \;+\; \dfrac{1}{12}\,t\,sin\,3t$

When $t = 10$,

$$\alpha \;=\; \frac{\pi}{3}\,cos\,30 \;+\; \frac{10}{12}\,sin\,30$$

$$=\; -\,0.662$$

[4 marks]

(c) As the question does not specify which side of the downward vertical the $\dfrac{\pi}{3}$ was, the sign of α is unimportant.

As 0.662 is close to 0.62 the model seems good when $t = 10$
(although that's only been checked for one value of t).

[1 mark]

(d) It's the $\dfrac{1}{2}\,cos\,3t$ that is forcing the motion so for simple harmonic motion that will need to be removed. So the obvious modification would be to switch to using the equation $\dfrac{d^2\theta}{dt^2} \;+\; 9\theta \;=\; 0$

[1 mark]