2.9 Answers

2.9.1 Solutions to 2.8 Exercise

Answer 1

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 10x = 0, \text{ when } t = 0,\ x = 0,\ \frac{dx}{dt} = -\frac{1}{4}$$

(i) Obtain the complementary function by solving the homogeneous equation,

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 10x = 0$$

$$m^2 + 2m + 10 = 0 \text{ is the auxiliary equation}$$

$$m = \frac{-2 \pm \sqrt{4 - 4 \times 10}}{2}$$

$$= \frac{-2 \pm \sqrt{(-1)(36)}}{2}$$

$$m = -1 \pm 3i \qquad \text{(complex roots)}$$

$\therefore$ The complementary function is $x = e^{-t}(A\cos 3t + B\sin 3t)$

Using the boundary conditions,

$x = e^{-t}(A\cos 3t + B\sin 3t) \qquad \Rightarrow 0 = A$

$\frac{dx}{dt} = e^{-t}(-3A\sin 3t + 3B\cos 3t) - e^{-t}(A\cos 3t + B\sin 3t)$

$\Rightarrow -\frac{1}{4} = (3B) - A \quad \therefore B = -\frac{1}{12}$

The particular solution is $x = -\frac{1}{12}e^{-t}\sin 3t$

[6 marks]

(ii) The motion is lightly damped because,

- there were two complex roots to the auxiliary equation.
- the graph shows that oscillations occurred.

[1 mark]

(iii) First, find the time at which the maximum displacement occurs.

$$x = -\frac{1}{12} e^{-t} \sin 3t$$

At maximum displacement, $\frac{dx}{dt} = 0$

$$-\frac{1}{12} e^{-t} 3\cos 3t + \frac{1}{12} e^{-t} \sin 3t = 0$$

Multiply through by $12\, e^{t}$

$$-3\cos 3t + \sin 3t = 0$$

$$\sin 3t = 3\cos 3t$$

$$\tan 3t = 3$$

$$3t = 1.249$$

$$t = 0.416 \text{ seconds}$$

$$x = -\frac{1}{12} e^{-0.416} \sin(3 \times 0.416)$$

$$= -0.0521 \text{ metres}$$

$$|x| = 5.21 \text{ centimetres}$$

[5 marks]

Answer 2

(a) Obtain the complementary function by solving the homogeneous equation,

$$4\frac{d^2h}{dt^2} + 4\frac{dh}{dt} + 37h = 0$$

$4m^2 + 4m + 37 = 0$ is the auxiliary equation

$$m = \frac{-4 \pm \sqrt{16 - 4 \times 4 \times 37}}{2 \times 4}$$

$$= \frac{-4 \pm \sqrt{(-1)(576)}}{8}$$

$$= \frac{-4 \pm 24\mathrm{i}}{8}$$

$$m = -\frac{1}{2} \pm 3\mathrm{i} \quad \text{(complex roots)}$$

$\therefore$ The complementary function is also the general solution in this case.

(As the particular integral is zero)

$$h = e^{-\frac{t}{2}}(A\cos 3t + B\sin 3t)$$

[2 marks]

(b) Boundary conditions; when $t = 0$, $h = -20$ and $\dfrac{dh}{dt} = 55$

$$h = e^{-\frac{t}{2}}(A\,cos\,3t + B\,sin\,3t) \Rightarrow -20 = A$$

$$\frac{dh}{dt} = e^{-\frac{t}{2}}(-3A\,sin\,3t + 3B\,cos\,3t) - \frac{1}{2}e^{-\frac{t}{2}}(A\,cos\,3t + B\,sin\,3t)$$

$$\Rightarrow 55 = (3B) - \frac{1}{2}A \quad \Rightarrow 55 = 3B + 10 \Rightarrow B = 15$$

The particular solution is $h = e^{-\frac{t}{2}}(-20\,cos\,3t + 15\,sin\,3t)$

At maximum displacement, $\dfrac{dx}{dt} = 0$

$$e^{-\frac{t}{2}}(60\,sin\,3t + 45\,cos\,3t) - \frac{1}{2}e^{-\frac{t}{2}}(-20\,cos\,3t + 15\,sin\,3t) = 0$$

$$120\,sin\,3t + 90\,cos\,3t + 20\,cos\,3t - 15\,sin\,3t = 0$$

$$105\,sin\,3t + 110\,cos\,3t = 0$$

$$tan\,3t = -\frac{110}{105}$$

$$tan\,3t = -\frac{22}{21}$$

$$3t = -0.8086,\ 2.333...$$

$$t = 0.778 \text{ seconds}$$

Displacement is then, $h = e^{-\frac{0.778}{2}}(-20\,cos\,2.333 + 15\,sin\,2.333)$

$$= 16.7\,\text{cm}$$

[8 marks]

(c) As t becomes large, the exponential term in the end-of-board displacement equation becomes small, and so h will also become small. From experience, this is how a diving board behaves once the diver has jumped off the end, an initially large oscillation fading reasonably quickly with the passage of time, returning to it's stationary equilibrium state.
(Technically the model has a negligible oscillation indefinitely)
I'd say it's a good model, for values of t between 0 and 10 seconds.
(If 10 seconds is what is meant by large)

[2 marks]

Answer 3

(a) The given boundary conditions are telling us that when $t = 0$,

$$\frac{d^2x}{dt^2} = 13.6, \qquad \frac{dx}{dt} = 0 \qquad \text{and} \qquad x = -20$$

$$5k\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + 17x = 0$$

$$\Rightarrow\ 5k\ (13.6) + 2k\ (0) + 17 \times (-20) = 0$$

$$68k = 340$$

$$k = 5$$

[2 marks]

(b) Obtain the complementary function by solving the homogeneous equation,

$$25\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 17x = 0$$

$$25\,m^2 + 10m + 17 = 0 \quad \text{is the auxiliary equation}$$

$$m = \frac{-10 \pm \sqrt{100 - 4 \times 25 \times 17}}{2 \times 25}$$

$$= \frac{-10 \pm \sqrt{(-1)(1600)}}{50}$$

$$m = -0.2 \pm 0.8\text{i} \qquad \text{(complex roots)}$$

$\therefore$ The complementary function is $x = e^{-0.2\,t}(\,A\cos 0.8t + B\sin 0.8t\,)$

Using the boundary conditions,

$x = e^{-0.2\,t}(\,A\cos 0.8t + B\sin 0.8t\,) \qquad \Rightarrow -20 = A$

$$\frac{dx}{dt} = e^{-0.2\,t}(-0.8A\sin 0.8t + 0.8B\cos 0.8t)$$

$$-\ 0.2e^{-0.2\,t}(A\cos 0.8t + B\sin 0.8t)$$

$$\Rightarrow\ 0 = (\,0.8\,B\,) - 0.2\,A$$

$$0.8\,B = 0.2\,A$$

$$B = -5$$

The particular solution is $x = e^{-0.2\,t}(-20\cos 0.8t - 5\sin 0.8t\,)$

[7 marks]

(c) $h = 30 + e^{-0.2\,\times 15}(-20\cos(0.8 \times 15) - 5\sin(0.8 \times 15\,)$

$= 29.3$ metres

[2 marks]

(d) It's unlikely that the rope will remain taught.
If it goes slack the tourist becomes a projectile for part of the motion.

[1 mark]

Answer 4

(i) α is the phase shift of the sine wave part of the expression but from the graph it is clear that there is no phase shift.
Alternatively, notice that when $t = 0$, $x = 0$ as the graph is at the origin.
$sin(\omega t + \alpha) = 0$ with $t = 0$ becomes $sin(\alpha) = 0 \therefore \alpha = 0$

[1 mark]

(ii) From the graph, $P = 12.5$ seconds and you may suspect this is 4π.

$$f = \frac{1}{P} = \frac{1}{4\pi} = 0.0796 \text{ Hertz (Cycles per second)}$$

$$P = \frac{2\pi}{\omega} \text{ so } \omega = \frac{2\pi}{P} = \frac{2\pi}{4\pi} = \frac{1}{2} \text{ radians/second}$$

[3 marks]

Answer 5

(a) Obtain the complementary function by solving the homogeneous equation,

$$100\frac{d^2x}{dt^2} + 60\frac{dx}{dt} + 13x = 0$$

$$100m^2 + 60m + 13 = 0 \quad \text{is the auxiliary equation}$$

$$m = \frac{-60 \pm \sqrt{3600 - 4 \times 100 \times 13}}{2 \times 100}$$

$$= \frac{-60 \pm \sqrt{(-1)(1600)}}{200}$$

$$= \frac{-60 \pm 40\text{i}}{200}$$

$$m = -0.3 \pm 0.2\text{i} \qquad \text{(complex roots)}$$

$\therefore$ The complementary function is $x = e^{-0.3t}(A\,cos\,0.2t + B\,sin\,0.2t)$

For the particular integral let $x = U$ in which case $\frac{d^2x}{dt^2} = \frac{dx}{dt} = 0$

Substituting this into $100\frac{d^2x}{dt^2} + 60\frac{dx}{dt} + 13x = 26$ gives,

$$13U = 26 \text{ so } U = 2$$

The general solution is $x = e^{-0.3t}(A\,cos\,0.2t + B\,sin\,0.2t) + 2$

[4 marks]

(b) Use the boundary conditions: when $t = 0$, $x = 0$ and $\dfrac{dx}{dt} = 10$

$$x = e^{-0.3t}(A\cos 0.2t + B\sin 0.2t) + 2 \quad \Rightarrow 0 = A + 2 \Rightarrow A = -2$$

$$\frac{dx}{dt} = e^{-0.3t}(-0.2A\sin 0.2t + 0.2B\cos 0.2t)$$

$$- 0.3e^{-0.3t}(A\cos 0.2t + B\sin 0.2t)$$

$$\Rightarrow 10 = 0.2B - 0.3 \times A$$

$$10 = 0.2B + 0.6$$

$$B = 47$$

The particular solution is $x = e^{-0.3t}(-2\cos 0.2t + 47\sin 0.2t) + 2$

At maximum concentration, $\dfrac{dx}{dt} = 0$

$$e^{-0.3t}(0.4\sin 0.2t + 9.4\cos 0.2t) - 0.3e^{-0.3t}(-2\cos 0.2t + 47\sin 0.2t) = 0$$

$$4\sin 0.2t + 94\cos 0.2t + 6\cos 0.2t - 141\sin 0.2t = 0$$

$$-137\sin 0.2t + 100\cos 0.2t = 0$$

$$\tan 0.2t = \frac{100}{137}$$

$$0.2t = 0.631, \ \ldots$$

$$t = 3.15 \text{ weeks}$$

Concentration is then,

$$x = e^{-0.3 \times 3.15}(-2\cos(0.2 \times 3.15) + 47\sin(0.2 \times 3.15) + 2$$

$$= 12.1\ \mu\text{g/ml}$$

[8 marks]

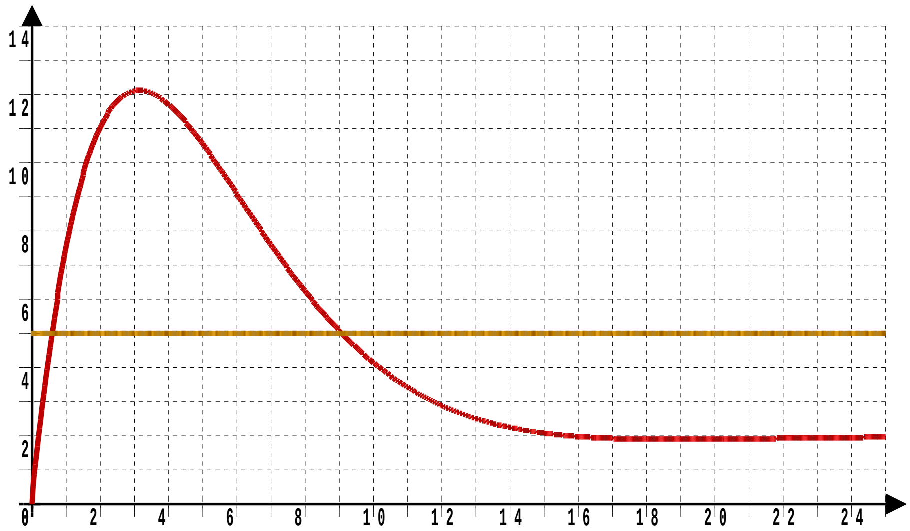
(c) Assume that any oscillations are negligible when $t = 10$

$$x = e^{-0.3 \times 10}(-2\cos(0.2 \times 10) + 47\sin(0.2 \times 10) + 2$$

$$= 4.17\,\mu\text{g/ml}$$

Which suggests it is safe to give the second dose after 10 weeks

A graph confirms the assumption about any oscillations being negligible after ten weeks.



[2 marks]

Answer 6

(i) The particle will exhibit (undamped) simple harmonic motion because

$\ddot{x} = -200\,x$ is of the form $\dfrac{d^2x}{dt^2} = -\omega^2 x$ with $\omega^2 = 200$

[1 mark]

(ii)

$$\frac{d^2x}{dt^2} + 200x = 0$$

$m^2 + 200 = 0$ is the auxiliary equation

$$m^2 - \left(10\sqrt{2}\,\mathrm{i}\right)^2 = 0$$

$m = \pm 10\sqrt{2}\,\mathrm{i}$ (complex roots)

The general solution is $x = \left(A\cos\left(10\sqrt{2}\,t\right) + B\sin\left(10\sqrt{2}\,t\right)\right)$

Using the $t = 0$ boundary conditions; $x = 0.3$ and the velocity is 0 $\mathrm{m.s^{-1}}$

$x = A\cos 3t + B\sin 3t \qquad \Rightarrow\ 0.3 = A$

$\dfrac{dx}{dt} = \left(-10\sqrt{2}\,A\sin\left(10\sqrt{2}\,t\right) + 10\sqrt{2}\,B\cos\left(10\sqrt{2}\,t\right)\right) \Rightarrow B = 0$

The particular solution is, $x = 0.3\cos\left(10\sqrt{2}\,t\right)$

[7 marks]

(iii) Period : $P = \dfrac{2\pi}{\omega}$

$$= \frac{2\pi}{10\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\pi\sqrt{2}}{10} \text{ seconds}$$

Amplitude : 0.3 metres

[3 marks]

(iv) $\dfrac{dx}{dt} = \left(-3\sqrt{2}\,\sin\left(10\sqrt{2}\,t\right)\right)$

$\therefore$ Maximum speed is $3\sqrt{2}$ $\mathrm{m.s^{-1}}$ (Approximately 4.24 $\mathrm{m.s^{-1}}$)

[2 marks]

Answer 7

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 2e^{-t}$$

(a) Obtain the complementary function by solving the homogeneous equation,

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0$$

$m^2 + 5m + 6 = 0$ is the auxiliary equation

$$(m + 3)(m + 2) = 0$$

Either $m = -3$ or $m = -2$ (two distinct real roots)

$\therefore$ The complementary function is $x = Ae^{-3t} + Be^{-2t}$

For a particular integral try $x = Ue^{-t}$

in which case $\frac{dx}{dt} = -Ue^{-t}$

and $\frac{d^2x}{dt^2} = Ue^{-t}$

Substitute these into the question's differential equation...

$$Ue^{-t} - 5Ue^{-t} + 6Ue^{-t} = 2e^{-t}$$

$$2Ue^{-t} = 2e^{-t}$$

$$U = 1$$

$\therefore$ The particular integral is $x = e^{-t}$

The general solution is $x = Ae^{-3t} + Be^{-2t} + e^{-t}$

Use the boundary conditions: when $t = 0$, $x = 0$ and $\frac{dx}{dt} = 2$

$x = Ae^{-3t} + Be^{-2t} + e^{-t} \quad \Rightarrow \quad 0 = A + B + 1$

$\frac{dx}{dt} = -3Ae^{-3t} - 2Be^{-3t} - e^{-t} \quad \Rightarrow \quad 2 = -3A - 2B - 1$

Solve these simultaneously to get $A = -1$ and $B = 0$

The particular solution is $x = -e^{-3t} + e^{-t}$

[8 marks]

(b) At a maximum (or minimum) distance

$$\frac{dx}{dt} = 0$$

$$3e^{-3t} - e^{-t} = 0$$

Multiply through by e^{3t}

$$
\begin{aligned}
3 - e^{2t} &= 0 \\
e^{2t} &= 3 \\
2t &= ln\,3 \\
t &= \frac{1}{2}\,ln\,3 \\
&= ln\sqrt{3} \text{ seconds}
\end{aligned}
$$

At this time,

$$
\begin{aligned}
x &= -e^{-3\,ln\sqrt{3}} + e^{-ln\sqrt{3}} \\
&= -e^{ln\sqrt{3}^{\,-3}} + e^{ln\sqrt{3}^{\,-1}} \\
&= -\sqrt{3}^{\,-3} + \sqrt{3}^{\,-1} \\
&= -\frac{1}{3\sqrt{3}} + \frac{1}{\sqrt{3}} \\
&= \frac{-1+3}{3\sqrt{3}} \\
&= \frac{2}{3\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \\
&= \frac{2\sqrt{3}}{9} \qquad \square
\end{aligned}
$$

To determine the nature of the turning point...

$$
\begin{aligned}
\frac{d^2x}{dt^2} &= -9\,e^{-3t} + e^{-t} \\
&= -9\,e^{-3\,ln\sqrt{3}} + e^{-ln\sqrt{3}} \quad \text{using } t = ln\sqrt{3} \\
&= -9\,e^{ln\sqrt{3}^{\,-3}} + e^{ln\sqrt{3}^{\,-1}} \\
&= -9\sqrt{3}^{\,-3} + \sqrt{3}^{\,-1} \\
&= -\frac{9}{3\sqrt{3}} + \frac{1}{\sqrt{3}} \\
&= -\frac{2\sqrt{3}}{3}
\end{aligned}
$$

As this is negative, x is indeed a maximum.

[7 marks]