3.4 Answers to 3.5 Exercise

Answer 1

(i) $\dfrac{dx}{dt} = -x + 6y$ $\qquad\qquad$ $\dfrac{dy}{dt} = x - 2y$

Work on first equation: $\dfrac{dx}{dt} = -x + 6y$

$$y = \frac{1}{6}\frac{dx}{dt} + \frac{1}{6}x \qquad \text{Making } y \text{ the subject}$$

$$\frac{dy}{dt} = \frac{1}{6}\frac{d^2x}{dt^2} + \frac{1}{6}\frac{dx}{dt} \qquad \text{By differentiation}$$

Work on second equation: $\dfrac{dy}{dt} = x - 2y$

$$\frac{1}{6}\frac{d^2x}{dt^2} + \frac{1}{6}\frac{dx}{dt} = x - 2\left(\frac{1}{6}\frac{dx}{dt} + \frac{1}{6}x\right)$$

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} = 6x - 2\frac{dx}{dt} - 2x$$

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} - 4x = 0$$

$$m^2 + 3m - 4 = 0$$

$$(m - 1)(m + 4) = 0$$

Either $m = 1$ or $m = -4$ (two distinct real roots)

$$x = Ae^{t} + Be^{-4t}$$

Differentiate this as we need a "$y =$" companion to this "$x =$"

$$\frac{dx}{dt} = Ae^{t} - 4Be^{-4t}$$

Go back to the first equation: $\dfrac{dx}{dt} = -x + 6y$

$$Ae^{t} - 4Be^{-4t} = -Ae^{t} - Be^{-4t} + 6y$$

$$6y = 2Ae^{t} - 3Be^{-4t}$$

$$y = \frac{1}{3}Ae^{t} - \frac{1}{2}Be^{-4t}$$

General Solutions: $x = Ae^{t} + Be^{-4t}$

$$y = \frac{1}{3}Ae^{t} - \frac{1}{2}Be^{-4t}$$

[8 marks]

(ii) Boundary conditions; $t = 0$, $x = 2$ and $y = 0$

$$x = Ae^{t} + Be^{-4t} \qquad \Rightarrow \; 2 = A + B$$

$$y = \frac{1}{3}Ae^{t} - \frac{1}{2}Be^{-4t} \qquad \Rightarrow \; 0 = \frac{1}{3}A - \frac{1}{2}B$$

Simultaneous equations,

$$2A + 2B = 4$$

SUBTRACT $\quad 2A - 3B = 0$

$$5B = 4$$

$$B = \frac{4}{5} \text{ and } A = \frac{6}{5}$$

The particular solutions are,

$$x = \frac{6}{5}e^{t} + \frac{4}{5}e^{-4t}$$

$$y = \frac{2}{5}e^{t} - \frac{2}{5}e^{-4t}$$

[4 marks]

Answer 2

(i) $\dfrac{dx}{dt} = -3x + 2y \qquad \dfrac{dy}{dt} = -2x + y$

Work on first equation: $\dfrac{dx}{dt} = -3x + 2y$

$$y = \frac{1}{2}\frac{dx}{dt} + \frac{3}{2}x \qquad \text{Making } y \text{ the subject}$$

$$\frac{dy}{dt} = \frac{1}{2}\frac{d^2x}{dt^2} + \frac{3}{2}\frac{dx}{dt} \qquad \text{By differentiation}$$

Work on second equation: $\dfrac{dy}{dt} = -2x + y$

$$\frac{1}{2}\frac{d^2x}{dt^2} + \frac{3}{2}\frac{dx}{dt} = -2x + \frac{1}{2}\frac{dx}{dt} + \frac{3}{2}x$$

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} = -4x + \frac{dx}{dt} + 3x$$

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = 0$$

$$m^2 + 2m + 1 = 0$$

$$(m + 1)^2 = 0$$

$$m = -1 \qquad \text{(repeated root)}$$

$$x = (A + Bt)\,e^{-t}$$

Differentiate this as we need a "$y =$" companion to this "$x =$"

$$\frac{dx}{dt} = (-1)(A + Bt)e^{-t} + Be^{-t}$$
$$= (B - A - Bt)e^{-t}$$

Go back to the first equation: $\frac{dx}{dt} = -3x + 2y$

$$(B - A - Bt)e^{-t} = -3(A + Bt)e^{-t} + 2y$$
$$2y = (B - A - Bt + 3A + 3Bt)e^{-t}$$
$$y = \frac{1}{2}(B + 2A + 2Bt)e^{-t}$$

General Solutions: $x = (A + Bt)e^{-t}$

$$y = \frac{1}{2}(B + 2A + 2Bt)e^{-t}$$

Boundary conditions; $t = 0$, $x = 100$ and $y = 200$

$x = (A + Bt)e^{-t} \quad \Rightarrow \quad 100 = A$

$y = \frac{1}{2}(B + 2A + 2Bt)e^{-t} \quad \Rightarrow \quad 200 = \frac{1}{2}B + A$ so $B = 200$

The particular solutions are,

$$x = 100(1 + 2t)e^{-t}$$
$$y = 200(1 + t)e^{-t}$$

[8 marks]

(ii) $x = 100(1 + 2 \times 2)e^{-2} = 67.7$ litres

$y = 200(1 + 2)e^{-2} = 81.2$ litres

[2 marks]

(iii) As t becomes large, e^{-t} tends towards zero.

So the amount of both chemicals will tend towards zero.

[2 marks]

Answer 3

(i) $\dfrac{dx}{dt} = 0.2\,x + 0.2\,y$ $\qquad$ $\dfrac{dy}{dt} = -\,0.5\,x + 0.4\,y$

Work on first equation: $\dfrac{dx}{dt} = 0.2\,x + 0.2\,y$

$$y = 5\,\frac{dx}{dt} - x \qquad \text{Making } y \text{ the subject}$$

$$\frac{dy}{dt} = 5\,\frac{d^2x}{dt^2} - \frac{dx}{dt} \qquad \text{By differentiation}$$

Work on second equation: $\dfrac{dy}{dt} = -\,0.5\,x + 0.4\,y$

$$5\,\frac{d^2x}{dt^2} - \frac{dx}{dt} = -\,0.5\,x + 0.4\left(5\,\frac{dx}{dt} - x\right)$$

$$5\,\frac{d^2x}{dt^2} - \frac{dx}{dt} = -\,0.5\,x + 2\,\frac{dx}{dt} - 0.4\,x$$

$$5\,\frac{d^2x}{dt^2} - 3\,\frac{dx}{dt} + 0.9x = 0$$

$$\frac{d^2x}{dt^2} - 0.6\,\frac{dx}{dt} + 0.18x = 0$$

[3 marks]

(ii) The auxiliary equation is,

$$m^2 - 0.6m + 0.18 = 0$$

$$100\,m^2 - 60m + 18 = 0$$

$$50\,m^2 - 30m + 9 = 0$$

$$m = \frac{30 \pm \sqrt{900 - 4 \times 50 \times 9}}{2 \times 50}$$

$$= \frac{30 \pm \sqrt{(-1)(900)}}{100}$$

$$= 0.3 \pm 0.3\mathrm{i}$$

General solution : $x = e^{0.3\,t}\,(A\,cos\,(0.3t) + B\,sin\,(0.3t))$

That is, $\alpha = \beta = 0.3$

[4 marks]

(iii) Differentiate part (ii) answer as we need a "$y =$" companion to this "$x =$"

$$\frac{dx}{dt} = e^{0.3t}(-0.3A\,sin(0.3t) + 0.3B\,cos(0.3t))$$

$$+ 0.3e^{0.3t}(A\,cos(0.3t) + B\,sin(0.3t))$$

$$= 0.3\,e^{0.3t}((B + A)\,cos(0.3t) + (B - A)\,sin(0.3t))$$

Go back to the first equation: $\frac{dx}{dt} = 0.2\,x + 0.2\,y$

$$y = 5\frac{dx}{dt} - x$$

$$y = 1.5\,e^{0.3t}((B + A)\,cos(0.3t) + (B - A)\,sin(0.3t)) - \left(e^{0.3t}(Acos(0.3t) + Bsin(0.3t))\right)$$

$$y = e^{0.3t}\{(1.5B + 1.5A - A)\,cos(0.3t) + (1.5B - 1.5A - B)\,sin(0.3t)\}$$

$$y = e^{0.3t}\{(0.5A + 1.5B)\,cos(0.3t) + (-1.5A + 0.5B)\,sin(0.3t)\}$$

$$y = 0.1\,e^{0.3t}\{(5A + 15B)\,cos(0.3t) + (-15A + 5B)\,sin(0.3t)\}$$

[3 marks]

(iv) Use the $t = 0$ conditions, $x = 3$ (Weasels), $y = 111$ (Foxes)

$$x = e^{0.3t}(A\,cos(0.3t) + B\,sin(0.3t)) \Rightarrow A = 3$$

$$y = 0.1\,e^{0.3t}\{(5A + 15B)\,cos(0.3t) + (-15A + 5B)\,sin(0.3t)\}$$

$$\Rightarrow 111 = 0.1(5A + 15B)$$

$$1110 = 15 + 15B$$

$$B = 73$$

Particular solutions: $x = e^{0.3t}(3\,cos(0.3t) + 73\,sin(0.3t))$

$$y = 0.1\,e^{0.3t}\{1110\,cos(0.3t) + 320\,sin(0.3t)\}$$

To find the time when $y = 0$?

$$1110\,cos(0.3t) + 320\,sin(0.3t) = 0$$

$$320\,sin(0.3t) = -1110\,cos(0.3t)$$

$$tan(0.3t) = -\frac{1110}{320}$$

$$= -3.469$$

$$0.3t = -1.290,\ 1.851,\ \ldots$$

$$t = 6.172 \text{ years}$$

Year Islands Foxes die out is $2023 + 6.172 =$ During 2029

[5 marks]

(v) $x = e^{0.3t}(3\cos(0.3t) + 73\sin(0.3t))$

The number of Weasels when $t = 6.172$ will be...

$x = e^{0.3 \times 6.172}(3\cos(0.3 \times 6.172) + 73\sin(0.3 \times 6.172))$

$= 441.5$

[1 mark]

(vi) The graph shows that the Island Foxes cannot be the only source of food for the Weasels, as their population goes on rising after the Island Foxes have died out. The question seemed to suggest that the populations of the two species were interrelated, but if this were so, the Island Foxes demise would have also caused the population of Weasels to suffer. This model is, perhaps, a bit too simple to realistically capture population dynamics.

[2 marks]