Further Pure A-Level Mathematics
Compulsory Course Component
Core 2

APPLICATIONS OF DIFFERENTIAL EQUATIONS

A pendulum can be a lot of fun !

The Sea Dragon at Fantasy Island Amusement Park, New Jersey, USA

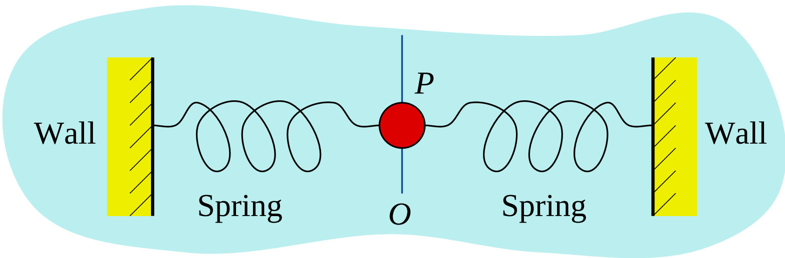
APPLICATIONS OF DIFFERENTIAL EQUATIONS

Lesson 1

Further A-Level Pure Mathematics, Core 2
Applications of Differential Equations

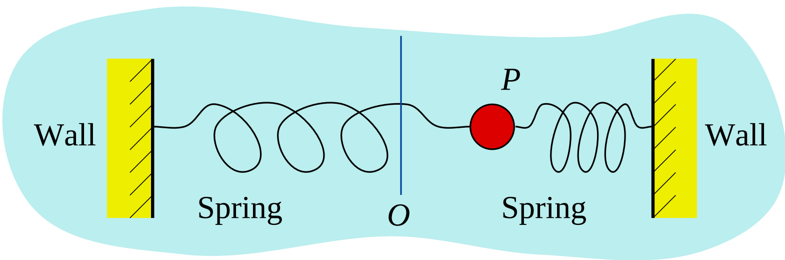
1.1 Simple Harmonic Motion

The classic mechanical example of a system that exhibits simple harmonic motion (SHM) is illustrated below.



Shown is a bird's eye view of a particle, P, resting on ice and sitting on the midway line, O, between a pair of parallel walls with two identical elastic springs attached.

The next illustration shows the particle after it has been moved and held to one side of O. It is now released and oscillates about the central position, O. This idealised oscillatory motion (that ignores any friction effects) is described as simple harmonic.



Simple harmonic motion is characterised by having an acceleration (towards O) that is proportional to the displacement of P from O. Written in mathematics this becomes $\dfrac{d^2x}{dt^2} = -\omega^2 x$ where ω^2 is the constant of the proportionality.

Physicists call ω the angular velocity of the particle. Note that the minus sign is there because the acceleration is always directed towards O, the centre of the oscillation. The above equation can be written as $\dfrac{d^2x}{dt^2} + \omega^2 x = 0$ which is a second order homogeneous differential equation. Number Wonder's Differential Equations II covers how to solve such equations.

1.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 60

Question 1

A particle is moving to and fro along a straight line.
At time t seconds its displacement in metres, x, from a fixed point O is such that,

$$\frac{d^2x}{dt^2} = -9x$$

At $t = 0$, $x = 1$ and the particle is moving with a velocity of 4 $m.s^{-1}$

(i) Is the motion of the particle simple harmonic ?

[1 mark]

(ii) By the use of standard techniques to obtaining the particular solution of a second order differential equation, find an expression for the displacement of the particle after t seconds.

[5 marks]

(iii) By writing your answer in the form $x = R\sin(\omega t + \alpha)$ determine the maximum displacement of the particle from O and the first time, correct to 3 significant figures, that the particle attains this maximum.

[4 marks]

Question 2

A particle P is moving along a straight line.
At time t seconds, the acceleration of the particle is given by

$$a = t + 5v, \; t \geqslant 0$$

(i) Show that the situation described can be modelled by the first order differential equation $\dfrac{dv}{dt} - 5v = t$

[1 mark]

(ii) Is the motion of the particle simple harmonic ?
Give a reason for your answer.

[2 marks]

(iii) State the integrating factor associated with the differential equation.

[1 mark]

(iv) Given that $v = 0$ when $t = 0$, show that the velocity of the particle at time t is given by an equation of the form $v = A(Be^{kt} - Ct - 1)$ where A, B, C and k are positive constants to be found.

[5 marks]

Question 3

A particle moves along a straight line. The particle moves such that its acceleration, in $m.s^{-2}$, acts towards a fixed point *O* and is proportional to its distance, *x* m, from *O*.

(i) Describe the motion of the particle.

[1 mark]

Given that the acceleration of the particle is -5 $m.s^{-2}$ when $x = 1$,

(ii) write down a differential equation to describe the motion of the particle.

[2 marks]

If the velocity and displacement of the particle at time $t = 0$ are 6 $m.s^{-1}$ and 5 m,

(iii) find an expression for the displacement of the particle after *t* seconds.

[7 marks]

(iv) Hence find the maximum distance of the particle from *O*.

[2 marks]

Question 4

Further A-Level Examination Question from June 2011, FP2, Q8 (Edexcel)

The differential equation $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = \cos 3t, \quad y \geqslant 0$

describes the motion of a particle along the x-axis.

(a) Find the general solution of this differential equation.

[8 marks]

(b) Find the particular solution for which, at $t = 0$, $x = \dfrac{1}{2}$ and $\dfrac{dx}{dt} = 0$

[5 marks]

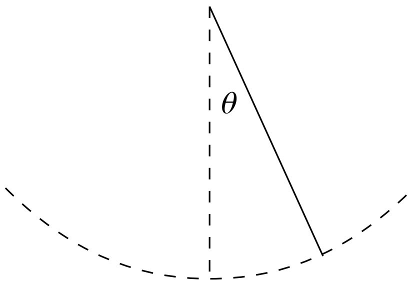
On the graph of the particular solution defined in part (b), the first turning point for $t > 30$ is the point A.

(c) Find approximate values for the coordinates of A.

[2 marks]

Question 5

Further A-Level Examination Question from June 2022, Paper 1, Q10 (Edexcel)



The motion of a pendulum is modelled by the differential equation,

$$\frac{d^2\theta}{dt^2} + 9\theta = \frac{1}{2}\cos 3t$$

where θ is the angle, in radians, that the pendulum makes with the downward vertical t seconds after it begins to move.

(a) **(i)** Show that a particular solution of the differential equation is,

$$\theta = \frac{1}{12}t\sin 3t$$

[4 marks]

(ii) Hence, find the general solution of the differential equation.

[4 marks]

Initially, the pendulum

- makes an angle of $\frac{\pi}{3}$ radians with the downward vertical
- is at rest

Given that, 10 seconds after it begins to move, the pendulum makes an angle of α radians with the downward vertical,

(b) determine, according to the model, the value of α to 3 significant figures.

[4 marks]

Given that the true value of α is 0.62

(c) evaluate the model.

[1 mark]

The differential equation $\frac{d^2\theta}{dt^2} + 9\theta = \frac{1}{2} cos\, 3t$ models the motion of the pendulum as moving with forced harmonic motion.

(d) Refine the differential equation so that the motion of the pendulum is simple harmonic motion.

[1 mark]

1.3 Answers

1.3.1 Solution to 1.2 Exercise

Answer 1

(i) Yes, because $\frac{d^2x}{dt^2} = -9x$ is of the form $\frac{d^2x}{dt^2} = -\omega^2 x$ with $\omega = 3$

which is the equation that characterises simple harmonic motion.

[1 mark]

(ii)

$$\frac{d^2x}{dt^2} + 9x = 0$$

$$m^2 + 9 = 0 \quad \text{is the auxiliary equation}$$

$$m^2 - (3\text{i})^2 = 0$$

$$m^2 = (3\text{i})^2$$

$$m = \pm 3\text{i} \qquad \text{(complex roots)}$$

The general solution is $x = e^{0t}(A\cos 3t + B\sin 3t)$

$$x = A\cos 3t + B\sin 3t$$

$$\frac{dx}{dt} = -3A\sin 3t + 3B\cos 3t$$

Using the boundary conditions, at $t = 0$, $x = 1$ and the velocity is 4 m.s^{-1}

$x = A\cos 3t + B\sin 3t \quad \Rightarrow \quad 1 = A$

$\frac{dx}{dt} = -3A\sin 3t + 3B\cos 3t \quad \Rightarrow \quad 4 = 3B$ so $B = \frac{4}{3}$

The particular solution is, $x = \cos 3t + \frac{4}{3}\sin 3t$

[5 marks]

(iii) In general, $x = R\sin(\omega t + \alpha) = R\sin\omega t\cos\alpha + R\cos\omega t\sin\alpha$

so $\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{1 \times 3}{4}$

$$\alpha = 0.644 \quad \text{and} \quad R = \sqrt{1^2 + \left(\frac{4}{3}\right)^2} = \frac{5}{3}$$

$x = \frac{5}{3}\sin(3t + 0.644)$ giving $x_{MAX} = \frac{5}{3}$ metres

First occurs when $\sin(3t + 0.644) = 1$, when, $3t + 0.644 = \frac{\pi}{2}$

Solve this to get $t = 0.309$ seconds.

[4 marks]

Answer 2

(i) $a = \dfrac{dv}{dt}$ and so $a = t + 5v$

becomes $\dfrac{dv}{dt} = t + 5v$

$$\therefore \ \frac{dv}{dt} - 5v = t \qquad \square$$

[1 mark]

(ii) $a = \dfrac{d^2x}{dt^2}$ and $v = \dfrac{dx}{dt}$ so $a = t + 5v$

can be recast as $\dfrac{d^2x}{dt^2} = t + 5\,\dfrac{dx}{dt}$

$$\frac{d^2x}{dt^2} - 5\,\frac{dx}{dt} = t$$

This is not in the form of the differential equation for SHM

[2 marks]

(iii) $\int P(x)\ dt = \int\ (-5)\ dt = -5t$ so integrating factor is e^{-5t}

[1 mark]

(iv)

$$\frac{dv}{dt} - 5v = t$$

Multiply through by the integrating factor

$$e^{-5t}\,\frac{dv}{dt} - 5\,e^{-5t}\,v = t\,e^{-5t}$$

integrate both sides with respect to t

$$e^{-5t}\,v = -\frac{1}{5}\int t\ (-5)\ e^{-5t}\,dt$$

$$= -\frac{1}{5}\left(t\,e^{-5t} - \int e^{-5t}\,dt + c_1 \right)$$

$$= -\frac{1}{25}\left(5t\,e^{-5t} + \int (-5)\ e^{-5t}\,dt + c_1 \right)$$

$$= -\frac{1}{25}\left(5t\,e^{-5t} + e^{-5t} + c_2 \right) +$$

$$v = \frac{1}{25}\left(-5t - 1 + c_3\,e^{5t} \right)$$

Use the boundary condition $v = 0$ when $t = 0$: $0 = \dfrac{1}{25}\left(0 - 1 + c_3 \right)$

$$c_3 = 1$$

$$\therefore\ v = \frac{1}{25}\left(e^{5t} - 5t - 1 \right)$$

[5 marks]

Answer 3

(i) It's simple harmonic motion.

[1 mark]

(ii) The equation is of the form $\frac{d^2x}{dt^2} = -\omega^2 x$

Using the information given, $-\ 5 = -\omega^2 \times 1$ so $\omega^2 = 5$

$$\therefore \ \frac{d^2x}{dt^2} = -5x$$

[2 marks]

(iii)

$$\frac{d^2x}{dt^2} + 5x = 0$$

$m^2 + 5 = 0$ is the auxiliary equation

$$m^2 - \left(\sqrt{5}\ \mathrm{i}\right)^2 = 0$$

$m = \pm\sqrt{5}\ \mathrm{i}$ (complex roots)

$$x = A\cos\sqrt{5}\ t + B\sin\sqrt{5}\ t$$

$$\frac{dx}{dt} = -\sqrt{5}\ A\sin\sqrt{5}\ t + \sqrt{5}\ B\cos\sqrt{5}\ t$$

$$\frac{d^2x}{dt^2} = -5A\cos\sqrt{5}\ t - 5B\sin\sqrt{5}\ t$$

Using the boundary conditions that at $t = 0$, $v = 6$ and $x = 5$

$x = A\cos\sqrt{5}\ t + B\sin\sqrt{5}\ t \Rightarrow 5 = A$

$\frac{dx}{dt} = -\sqrt{5}\ A\sin\sqrt{5}\ t + \sqrt{5}\ B\cos\sqrt{5}\ t \Rightarrow 6 = \sqrt{5}\ B$ so $B = \frac{6\sqrt{5}}{5}$

$$\therefore \ x = 5\cos\sqrt{5}\ t + \frac{6\sqrt{5}}{5}\ \sin\sqrt{5}\ t$$

[7 marks]

(iv) $D_{MAX} = \sqrt{5^2 + \left(\frac{6\sqrt{5}}{5}\right)^2}$

$= \frac{\sqrt{805}}{5}$

[2 marks]

Answer 4

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = \cos 3t$$

(a) Obtain the complementary function by solving the homogeneous equation,

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = 0$$

$m^2 + 6m + 9 = 0$ is the auxiliary equation

$$(m + 3)^2 = 0$$

$m = -3$ (repeated root)

$\therefore$ The complementary function is $x = (A + Bx)\, e^{-3t}$

For a particular integral try $x = U\cos 3t + V\sin 3t$

in which case $\dfrac{dx}{dt} = -3U\sin 3t + 3V\cos 3t$

and $\dfrac{d^2x}{dt^2} = -9U\cos 3t - 9V\sin 3t$

Substitute these into the question's differential equation...

$$-9U\cos 3t - 9V\sin 3t - 18U\sin 3t + 18V\cos 3t + 9U\cos 3t + 9V\sin 3t = \cos 3t$$

$$(-9U + 18V + 9U)\cos 3t + (-9V - 18U + 9V)\sin 3t = \cos 3t$$

$$18V = 1 \text{ so } V = \frac{1}{18}$$

$$-18U = 0 \text{ so } U = 0$$

$\therefore$ The particular integral is $x = \dfrac{1}{18}\sin 3t$

The general solution is $x = (A + Bt)\, e^{-3t} + \dfrac{1}{18}\sin 3t$

[8 marks]

(b) $\dfrac{dx}{dt} = (A + Bt)(-3)e^{-3t} + Be^{-3t} + \dfrac{1}{6}\cos 3t$

Use the boundary conditions: when $t = 0,\ x = \dfrac{1}{2}$ and $\dfrac{dx}{dt} = 0$

$x = (A + Bt)\, e^{-3t} + \dfrac{1}{18}\sin 3t \Rightarrow A = \dfrac{1}{2}$

$\dfrac{dx}{dt} = (A + Bt)(-3)e^{-3t} + Be^{-3t} + \dfrac{1}{6}\cos 3t$

$$\Rightarrow -\frac{3}{2} + B + \frac{1}{6} = 0 \text{ so } B = \frac{4}{3}$$

The particular solution is $x = \left(\dfrac{1}{2} + \dfrac{4}{3}t\right) e^{-3t} + \dfrac{1}{18}\sin 3t$

[5 marks]

(c) From $\dfrac{dx}{dt} = (A + Bt)(-3)e^{-3t} + Be^{-3t} + \dfrac{1}{6}\cos 3t$

with the values of A and B just worked out,

$$\frac{dx}{dt} = \left(\frac{1}{2} + \frac{4}{3}t\right)(-3)e^{-3t} + \frac{4}{3}e^{-3t} + \frac{1}{6}\cos 3t$$

For a turning point, we require,

$$\left(\frac{1}{2} + \frac{4}{3}t\right)(-3)e^{-3t} + \frac{4}{3}e^{-3t} + \frac{1}{6}\cos 3t = 0$$

For $t > 30$, the terms involving e^{-3t} will be negligible.

$$\therefore \text{ Solve } \quad \cos 3t = 0$$

$$3t = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots, \frac{(2n-1)\pi}{2}, \ldots$$

$$t = \frac{\pi}{6}, \frac{3\pi}{6}, \ldots, \frac{(2n-1)\pi}{6}, \ldots$$

We want $\dfrac{(2n-1)\pi}{6} > 30$

$$2n - 1 > \frac{30 \times 6}{\pi}$$

$$2n - 1 > 57.3$$

$$n > 29.1 \Rightarrow n = 30 \text{ (must be an integer)}$$

$$t = \frac{(2 \times 30 - 1)\pi}{6}$$

$$= \frac{59\pi}{6}$$

$$x = \frac{1}{18}\sin\left(\frac{3 \times 59\pi}{6}\right)$$

$$= -\frac{1}{18}$$

The coordinates of A are thus $\left(\dfrac{59\pi}{6}, -\dfrac{1}{18}\right)$

[2 marks]

Answer 5

(a) **(i)**

$$\theta = \frac{1}{12} t \sin 3t$$

$$\frac{d\theta}{dt} = \frac{1}{4} t \cos 3t + \frac{1}{12} \sin 3t$$

$$\frac{d^2\theta}{dt^2} = -\frac{3}{4} t \sin 3t + \frac{1}{4} \cos 3t + \frac{1}{4} \cos 3t$$

$$= \frac{1}{2} \cos 3t - \frac{3}{4} t \sin 3t$$

Now to show that these bits satisfy the given equation...

$$\text{LHS} = \frac{d^2\theta}{dt^2} + 9\theta$$

$$= \frac{1}{2} \cos 3t - \frac{3}{4} t \sin 3t + 9\left(\frac{1}{12} t \sin 3t\right)$$

$$= \frac{1}{2} \cos 3t - \frac{3}{4} t \sin 3t + \frac{3}{4} t \sin 3t$$

$$= \frac{1}{2} \cos 3t$$

$$= \text{RHS} \qquad \square$$

So the particular integral is indeed $\theta = \frac{1}{12} t \sin 3t$

[4 marks]

(ii) Obtaining the complementary function...

$$\frac{d^2\theta}{dt^2} + 9\theta = 0 \quad \text{homogeneous equation}$$

$$m^2 + 9 = 0 \quad \text{is the auxiliary equation}$$

$$m^2 - (3\text{i})^2 = 0$$

$$m = \pm 3\text{i} \quad \text{(complex roots)}$$

$\therefore$ The complementary function is $\theta = A \cos 3t + B \sin 3t$

General solution: $\theta = A \cos 3t + B \sin 3t + \frac{1}{12} t \sin 3t$

[4 marks]

(b) Now to use the boundary conditions, when $t = 0$, $\theta = \dfrac{\pi}{3}$ and $\dfrac{d\theta}{dt} = 0$

$$\theta = A\cos 3t + B\sin 3t + \frac{1}{12}t\sin 3t \Rightarrow A = \frac{\pi}{3}$$

$$\frac{d\theta}{dt} = -3A\sin 3t + 3B\cos 3t + \frac{3}{12}t\cos 3t - \frac{1}{12}\sin 3t \Rightarrow B = 0$$

Particular solution: $\theta = \dfrac{\pi}{3}\cos 3t + \dfrac{1}{12}t\sin 3t$

When $t = 10$,

$$\alpha = \frac{\pi}{3}\cos 30 + \frac{10}{12}\sin 30$$
$$= -0.662$$

[4 marks]

(c) As the question does not specify which side of the downward vertical the $\dfrac{\pi}{3}$ was, the sign of α is unimportant.

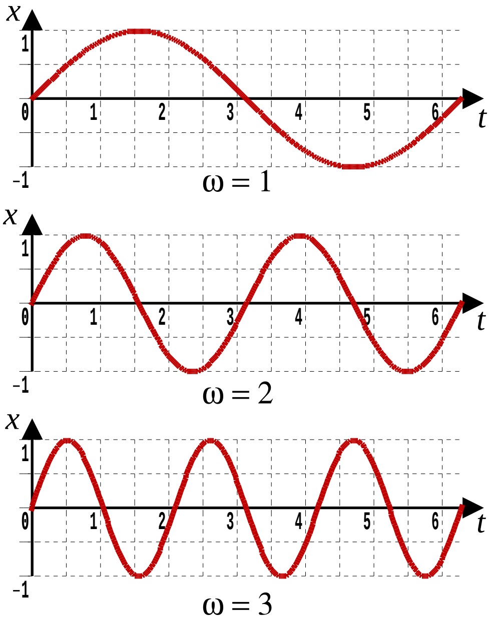
As 0.662 is close to 0.62 the model seems good when $t = 10$
(although that's only been checked for one value of t).

[1 mark]

(d) It's the $\dfrac{1}{2}\cos 3t$ that is forcing the motion so for simple harmonic motion that will need to be removed. So the obvious modification would be to switch to using the equation $\dfrac{d^2\theta}{dt^2} + 9\theta = 0$

[1 mark]

2.1 Period and Frequency



The graphs above are of the oscillations $y = \sin(\omega t)$ for $\omega = 1$, $\omega = 2$ and $\omega = 3$.

The value of ω gives the number of oscillations within 2π radians of time.

Definition : Period of a Function

The period, P, of an oscillatory function is the amount of time for the graph to complete a single oscillation.

$$P = \frac{2\pi}{\omega}$$

Example

So for the above three graphs the periods are,

$$\text{for } y = \sin t \text{ the period is } \frac{2\pi}{1} = 2\pi$$

$$\text{for } y = \sin 2t \text{ the period is } \frac{2\pi}{2} = \pi$$

$$\text{for } y = \sin 3t \text{ the period is } \frac{2\pi}{3}$$

Definition : Frequency of a Function

The frequency, f, of an oscillation is the number of vibrations per second.

$$f = \frac{1}{P} \quad \text{where } P \text{ is the period}$$

2.2 Forced Harmonic Motion

Simple harmonic motion is characterised by the differential equation,

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

and the frequency associated with this equation is that with which the system it describes would naturally vibrate. This is termed the system's resonant frequency. A system can be made to vibrate at a rate that is different to its natural rate in which case the above equation may take the form,

$$\frac{d^2x}{dt^2} + \omega^2 x = f(t)$$

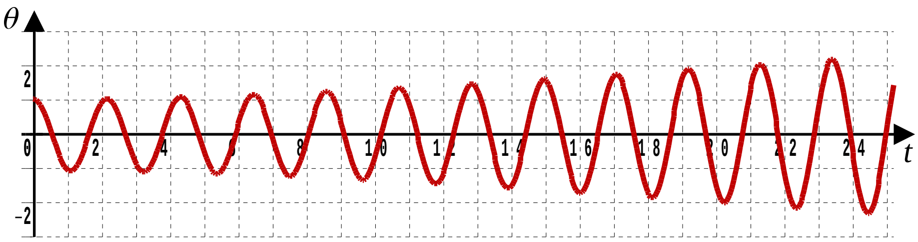
This is termed forced harmonic motion and the last question of the previous lesson (Exercise 1.2, Question 5) featured a system undergoing forced harmonic motion. The system being studied was that of a simple pendulum swinging from side to side where θ was the amount of angle, in radians, away from the vertical downward as time varied. The equation looked at was,

$$\frac{d^2\theta}{dt^2} + 9\theta = \frac{1}{2}\cos 3t$$

This would naturally vibrate at a rate of one third of an oscillation per second and the waveform would be a (phase shifted) sine wave. In this case the forcing term was in tune with the natural rate and so the waveform remained essentially a (phase shifted) sine wave but the forcing term is feeding in extra energy into the system and so the oscillations are increasing in amplitude. In fact, in answering the question it was worked out that,

$$\theta = \frac{\pi}{3}\cos 3t + \frac{1}{12}t\sin 3t$$

This is graphed below and the main feature to note is that the swing of the pendulum is steadily increasing. This is a system that may eventually break !



2.3 Damped Harmonic Motion

The foregoing illustrated a system that would eventually shake itself apart; an undesirable property ! The model for simple harmonic motion can be refined by adding an additional force which is proportional to the velocity of the particle. When this force acts such that it slows the particle it is known as a damping force and the motion of the particle is termed damped harmonic motion. This will have a second order differential equation of the form,

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + \omega^2 x = 0 \quad \text{for positive constants } k \text{ and } \omega^2$$

where x is the displacement from a fixed point at time, t.

The nature of the damping is classified as being one of three types, depending on the discriminant, D, of the auxiliary equation.

$\boldsymbol{D > 0}$: two distinct real roots; the system experiences **heavy damping**
The system does not oscillate. Motion is curtailed as if moving through treacle.

$\boldsymbol{D = 0}$: one repeated root; the system experiences **critical damping**
The system does not oscillate and motion is briskly curtailed.

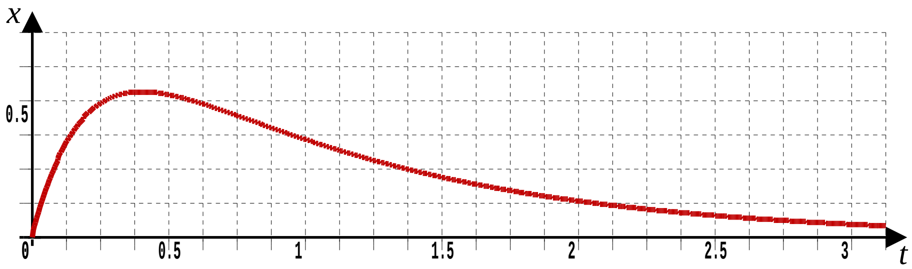
$\boldsymbol{D < 0}$: two complex roots; the system experiences **light damping**
The system oscillates with an amplitude that decreases exponentially over time.

2.4 Example of Heavy Damping

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 5x = 0 \quad \text{and when } t = 0,\ x = 0 \text{ and } \frac{dx}{dt} = 4$$

This has two distinct roots and so is an example of heavy damping.

This has general solution, $x = e^{-t} - e^{-5t}$ graphed below.



The push-tap in many motorway service stations, restaurants and schools is an example of a heavily damped mechanism. The tap is pressed and water flows for around half a minute whilst the tap slowly returns by itself to the off position.

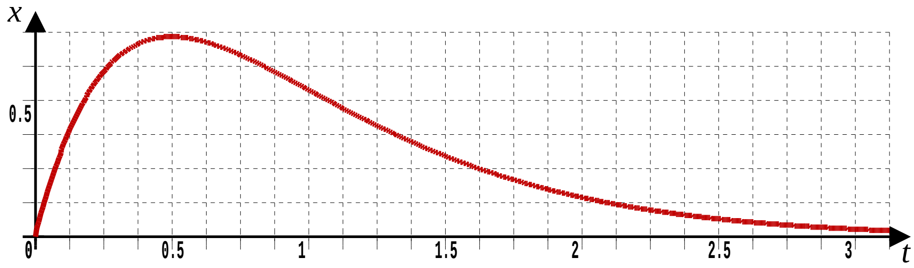
Their main advantage is that they can not be left running to overflow the basin either accidentally or by anti-social behaviourists.

2.5 Example of Critical Damping

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = 0 \quad \text{and when } t = 0,\ x = 0 \text{ and } \frac{dx}{dt} = 4$$

This has a repeated root and so is an example of critical damping.

This has general solution $x = 4t\,e^{-2t}$ graphed below.



A door closer is an example of a critically damped system. For fire safety and energy conservation reasons, the door should automatically close in as short a time as possible without slamming.

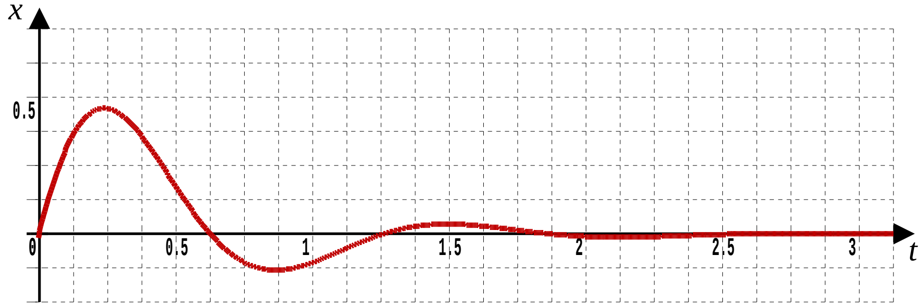
A ship's compass is another example of a critically damped device. When the ship changes direction the compass has to point in the new direction as quickly as possible without oscillating about the new heading. The damping is provided by filling the plastic sphere housing the device with clear oil.

2.6 Example of Light Damping

$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 29x = 0$ and when $t = 0,\ x = 0$ and $\frac{dx}{dt} = 4$

This has complex roots and so is an example of light damping.

This has general solution $x = \frac{4}{5} e^{-2t} \sin 5t$



If an electric light, handing from a ceiling by a wire, is accidentally knocked it will swing to and fro for a while. Slowly, over time, it will swing with less and less amplitude until is is again stationary. This is indeed "light" damping !

2.7 Undamped Simple Harmonic Motion

What is often referred to as simply "simple harmonic motion", characterised by,

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

is **undamped** simple harmonic motion.

The amplitude of its oscillations remain constant over time.

2.8 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 84

Question 1

Caspian × Herring Gull photograph by Matt Livesey, December 2019

A seagull is resting in calm weather conditions on a buoy on a lake.
When disturbed, it flies off.
The subsequent motion of the buoy is modelled by the differential equation,

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 10x = 0$$

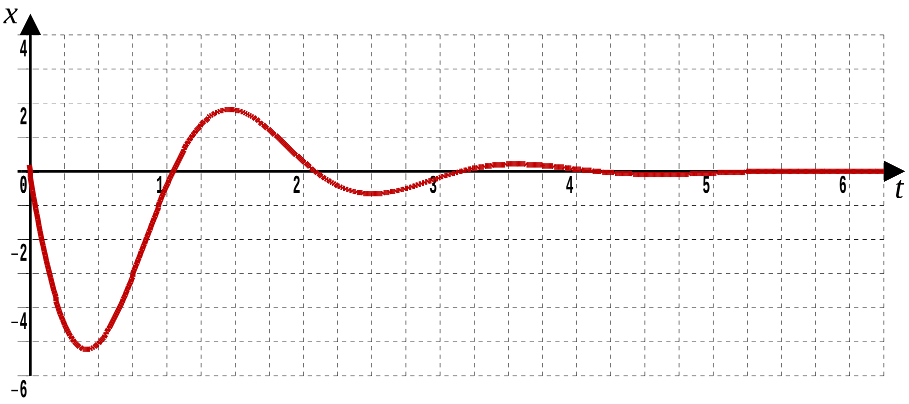
where t is time measured in seconds, and x is the vertical displacement, in metres, of the waterline of the buoy above or below the surface of the lake (up is positive).

When $t = 0$, $x = 0$ and, from the force of the seagull taking off, $\frac{dx}{dt} = -\frac{1}{4}$ m.s^{-1}

(i) Find x as a function of t

[6 marks]

A graph of the buoy's motion is given below, with displacement, x, given in centimetres (rather than metres) and time, t, in seconds.



(ii) Is the motion heavily, critically or lightly damped ?
Give a reason for your answer.

[1 mark]

(iii) Find the magnitude of the maximum displacement under the lake's surface of the buoy's waterline during the motion.
Give your answer in centimetres to one decimal place.

[5 marks]

Question 2

Further A-Level Examination Question from June 2019, Paper 2, Q6 (Edexcel)

An engineer is investigating the motion of a sprung diving board at a swimming pool. Let E be the position of the end of the diving board when it is at rest in its equilibrium position and when there is no diver standing on the diving board.

A diver jumps from the diving board.

The vertical displacement, h cm, of the end of the diving board above E is modelled by the differential equation,

$$4\frac{d^2h}{dt^2} + 4\frac{dh}{dt} + 37h = 0$$

where t seconds is the time after the diver jumps.

(a) Find a general solution of the differential equation.

[2 marks]

When $t = 0$, the end of the diving board is 20 cm below E and is moving upwards with a speed of 55 cm.s^{-1}

(b) Find, according to the model, the maximum vertical displacement of the end of the diving board above E.

[8 marks]

(c) Comment on the suitability of the model for large values of t

[2 marks]

Question 3

Further A-Level Examination Question from June 2021, Paper 1, Q6 (Edexcel)

A tourist decides to do a bungee jump from a bridge over a river.

One end of an elastic rope is attached to the bridge and the other end of the elastic rope is attached to the tourist.

The tourist jumps off the bridge.

At time t seconds after the tourist reaches their lowest point, their vertical displacement is x metres above a fixed point 30 metres vertically above the river.

When $t = 0$

- $x = -20$
- the velocity of the tourist is 0 $m.s^{-1}$
- the acceleration of the tourist is 13.6 $m.s^{-2}$

In the subsequent motion, the elastic rope is assumed to remain taut so that the vertical displacement of the tourist can be modelled by the differential equation

$5k\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + 17x = 0$ for $t \geqslant 0$ and where k is a positive constant.

(a) Determine the value of k

[2 marks]

(b) Determine the particular solution to the differential equation.

[7 marks]

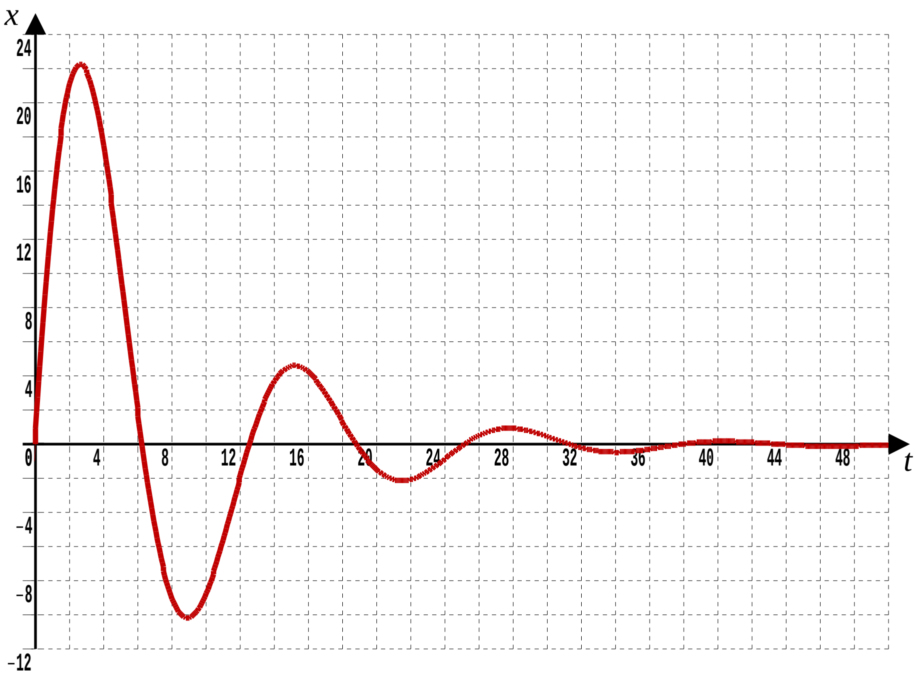
(c) Hence find, according to the model, the vertical height of the tourist above the river 15 seconds after they have reached their lowest point.

[2 marks]

(d) Give a limitation of the model.

[1 mark]

Question 4



Graphed is lightly damped harmonic motion of the form $x = Ae^{-kt} sin(\omega t + \alpha)$ with displacement, *x*, in metres and time, *t*, in seconds.

(i) Explain why $\alpha = 0$ radians.

[1 mark]

(ii) What is the period, *P*, the frequency, *f*, and the angular velocity, ω, of the waveform ?

[3 marks]

Question 5

Further A-Level Examination Question from May 2020, Paper 1, Q3 (Edexcel)

A scientist is investigating the concentration of antibodies in the bloodstream of a patient following a vaccination. The concentration of antibodies, x, measured in micrograms (μg) per millilitre (ml) of blood, is modelled by the differential equation $100\dfrac{d^2x}{dt^2} + 60\dfrac{dx}{dt} + 13x = 26$

where t is the number of weeks since the vaccination was given.

(a) Find a general solution of the differential equation.

[4 marks]

Initially,

- there are no antibodies in the bloodstream of the patient
- the concentration of antibodies is estimated to be increasing at 10 μg/ml per week

(b) Find, according to the model, the maximum concentration of antibodies in the bloodstream of the patient after the vaccination.

[8 marks]

A second dose of the vaccine has to be given to try to ensure that it is effective. It is only safe to give the second dose if the concentration of the antibodies in the bloodstream of the patient is less than 5 μg/ml.

(c) Determine whether, according to the model, it is safe to give the second dose of the vaccine to the patient exactly 10 weeks after the first dose.

[2 marks]

Question 6

A particle P is attached to one end of a light elastic spring. The other end of the spring is fixed to a point A on the smooth horizontal surface on which P rests. The particle is held at rest with AP = 0.9 metres and then released. At time t seconds the displacement of the particle from A is x metres. The motion of the particle can be modelled using the equation,

$$\ddot{x} = -200x$$

(i) State the type of motion exhibited by the particle P.

[1 mark]

Given that $x = 0.3$ and the particle is at rest when $t = 0$

(ii) solve the differential equation to find x as a function of t

[7 marks]

(iii) find the period and amplitude of the motion

[3 marks]

(iv) calculate the maximum speed of *P*

[2 marks]

Question 7

Further A-Level Examination Question from June 2009, FP2, Q8 (Edexcel)

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 2e^{-t}$$

Given that $x = 0$ and $\frac{dx}{dt} = 2$ at $t = 0$,

(a) find x in terms of t

[8 marks]

The solution to part (a) is used to represent the motion of a particle P on the x-axis. At time t seconds, where $t > 0$, P is x metres from the origin O.

(b) Show that the maximum distance between O and P is $\dfrac{2\sqrt{3}}{9}$ metres and justify that this distance is a maximum.

[7 marks]

2.9 Answers

2.9.1 Solutions to 2.8 Exercise

Answer 1

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 10x = 0, \text{ when } t = 0,\ x = 0,\ \frac{dx}{dt} = -\frac{1}{4}$$

(i) Obtain the complementary function by solving the homogeneous equation,

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 10x = 0$$

$$m^2 + 2m + 10 = 0 \quad \text{is the auxiliary equation}$$

$$m = \frac{-2 \pm \sqrt{4 - 4 \times 10}}{2}$$

$$= \frac{-2 \pm \sqrt{(-1)(36)}}{2}$$

$$m = -1 \pm 3\mathrm{i} \qquad \text{(complex roots)}$$

$\therefore$ The complementary function is $x = e^{-t}(A\cos 3t + B\sin 3t)$

Using the boundary conditions,

$x = e^{-t}(A\cos 3t + B\sin 3t) \qquad \Rightarrow \ 0 = A$

$$\frac{dx}{dt} = e^{-t}(-3A\sin 3t + 3B\cos 3t) - e^{-t}(A\cos 3t + B\sin 3t)$$

$$\Rightarrow -\frac{1}{4} = (3B) - A \quad \therefore B = -\frac{1}{12}$$

The particular solution is $x = -\dfrac{1}{12}e^{-t}\sin 3t$

[6 marks]

(ii) The motion is lightly damped because,

- there were two complex roots to the auxiliary equation.
- the graph shows that oscillations occurred.

[1 mark]

(iii) First, find the time at which the maximum displacement occurs.

$$x = -\frac{1}{12} e^{-t} \sin 3t$$

At maximum displacement, $\dfrac{dx}{dt} = 0$

$$-\frac{1}{12} e^{-t}\, 3\cos 3t + \frac{1}{12} e^{-t} \sin 3t = 0$$

Multiply through by $12\, e^{t}$

$$-3\cos 3t + \sin 3t = 0$$

$$\sin 3t = 3\cos 3t$$

$$\tan 3t = 3$$

$$3t = 1.249$$

$$t = 0.416 \text{ seconds}$$

$$x = -\frac{1}{12} e^{-0.416} \sin\,(3 \times 0.416)$$

$$= -0.0521 \text{ metres}$$

$$|x| = 5.21 \text{ centimetres}$$

[5 marks]

Answer 2

(a) Obtain the complementary function by solving the homogeneous equation,

$$4\frac{d^2h}{dt^2} + 4\frac{dh}{dt} + 37h = 0$$

$4m^2 + 4m + 37 = 0$ is the auxiliary equation

$$m = \frac{-4 \pm \sqrt{16 - 4 \times 4 \times 37}}{2 \times 4}$$

$$= \frac{-4 \pm \sqrt{(-1)(576)}}{8}$$

$$= \frac{-4 \pm 24\mathrm{i}}{8}$$

$$m = -\frac{1}{2} \pm 3\mathrm{i} \qquad \text{(complex roots)}$$

$\therefore$ The complementary function is also the general solution in this case.

(As the particular integral is zero)

$$h = e^{-\frac{t}{2}}\,(A\cos 3t + B\sin 3t)$$

[2 marks]

(b) Boundary conditions; when $t = 0$, $h = -20$ and $\dfrac{dh}{dt} = 55$

$$h = e^{-\frac{t}{2}}(A\cos 3t + B\sin 3t) \Rightarrow -20 = A$$

$$\frac{dh}{dt} = e^{-\frac{t}{2}}(-3A\sin 3t + 3B\cos 3t) - \frac{1}{2}e^{-\frac{t}{2}}(A\cos 3t + B\sin 3t)$$

$$\Rightarrow 55 = (3B) - \frac{1}{2}A \quad \Rightarrow 55 = 3B + 10 \Rightarrow B = 15$$

The particular solution is $h = e^{-\frac{t}{2}}(-20\cos 3t + 15\sin 3t)$

At maximum displacement, $\dfrac{dx}{dt} = 0$

$$e^{-\frac{t}{2}}(60\sin 3t + 45\cos 3t) - \frac{1}{2}e^{-\frac{t}{2}}(-20\cos 3t + 15\sin 3t) = 0$$

$$120\sin 3t + 90\cos 3t + 20\cos 3t - 15\sin 3t = 0$$

$$105\sin 3t + 110\cos 3t = 0$$

$$\tan 3t = -\frac{110}{105}$$

$$\tan 3t = -\frac{22}{21}$$

$$3t = -0.8086,\ 2.333...$$

$$t = 0.778 \text{ seconds}$$

Displacement is then, $h = e^{-\frac{0.778}{2}}(-20\cos 2.333 + 15\sin 2.333)$

$$= 16.7 \text{ cm}$$

[8 marks]

(c) As t becomes large, the exponential term in the end-of-board displacement equation becomes small, and so h will also become small. From experience, this is how a diving board behaves once the diver has jumped off the end, an initially large oscillation fading reasonably quickly with the passage of time, returning to it's stationary equilibrium state.
(Technically the model has a negligible oscillation indefinitely)
I'd say it's a good model, for values of t between 0 and 10 seconds.
(If 10 seconds is what is meant by large)

[2 marks]

Answer 3

(a) The given boundary conditions are telling us that when $t = 0$,

$$\frac{d^2x}{dt^2} = 13.6, \qquad \frac{dx}{dt} = 0 \qquad \text{and} \qquad x = -20$$

$$5k\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + 17x = 0$$

$$\Rightarrow 5k\ (13.6) + 2k\ (0) + 17 \times (-20) = 0$$

$$68k = 340$$

$$k = 5$$

[2 marks]

(b) Obtain the complementary function by solving the homogeneous equation,

$$25\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 17x = 0$$

$$25\,m^2 + 10m + 17 = 0 \quad \text{is the auxiliary equation}$$

$$m = \frac{-10 \pm \sqrt{100 - 4 \times 25 \times 17}}{2 \times 25}$$

$$= \frac{-10 \pm \sqrt{(-1)(1600)}}{50}$$

$$m = -0.2 \pm 0.8\mathrm{i} \qquad \text{(complex roots)}$$

$\therefore$ The complementary function is $x = e^{-0.2\,t}(\,A\cos 0.8t + B\sin 0.8t\,)$

Using the boundary conditions,

$x = e^{-0.2\,t}(\,A\cos 0.8t + B\sin 0.8t\,) \qquad \Rightarrow -20 = A$

$$\frac{dx}{dt} = e^{-0.2\,t}(-0.8A\sin 0.8t + 0.8B\cos 0.8t)$$

$$-\ 0.2e^{-0.2\,t}(A\cos 0.8t + B\sin 0.8t)$$

$$\Rightarrow 0 = (\,0.8\,B\,) - 0.2\,A$$

$$0.8\,B = 0.2\,A$$

$$B = -5$$

The particular solution is $x = e^{-0.2\,t}(-20\cos 0.8t - 5\sin 0.8t\,)$

[7 marks]

(c) $h = 30 + e^{-0.2\,\times 15}(-20\cos(0.8 \times 15) - 5\sin(0.8 \times 15\,)$

$= 29.3$ metres

[2 marks]

(d) It's unlikely that the rope will remain taught.
If it goes slack the tourist becomes a projectile for part of the motion.

[1 mark]

Answer 4

(i) α is the phase shift of the sine wave part of the expression but from the graph it is clear that there is no phase shift.

Alternatively, notice that when $t = 0$, $x = 0$ as the graph is at the origin.

$sin(\omega t + \alpha) = 0$ with $t = 0$ becomes $sin(\alpha) = 0 \therefore \alpha = 0$

[1 mark]

(ii) From the graph, $P = 12.5$ seconds and you may suspect this is 4π.

$$f = \frac{1}{P} = \frac{1}{4\pi} = 0.0796 \text{ Hertz (Cycles per second)}$$

$$P = \frac{2\pi}{\omega} \text{ so } \omega = \frac{2\pi}{P} = \frac{2\pi}{4\pi} = \frac{1}{2} \text{ radians/second}$$

[3 marks]

Answer 5

(a) Obtain the complementary function by solving the homogeneous equation,

$$100\frac{d^2x}{dt^2} + 60\frac{dx}{dt} + 13x = 0$$

$$100m^2 + 60m + 13 = 0 \quad \text{is the auxiliary equation}$$

$$m = \frac{-60 \pm \sqrt{3600 - 4 \times 100 \times 13}}{2 \times 100}$$

$$= \frac{-60 \pm \sqrt{(-1)(1600)}}{200}$$

$$= \frac{-60 \pm 40i}{200}$$

$$m = -0.3 \pm 0.2i \qquad \text{(complex roots)}$$

$\therefore$ The complementary function is $x = e^{-0.3t}(A\cos 0.2t + B\sin 0.2t)$

For the particular integral let $x = U$ in which case $\frac{d^2x}{dt^2} = \frac{dx}{dt} = 0$

Substituting this into $100\frac{d^2x}{dt^2} + 60\frac{dx}{dt} + 13x = 26$ gives,

$$13U = 26 \text{ so } U = 2$$

The general solution is $x = e^{-0.3t}(A\cos 0.2t + B\sin 0.2t) + 2$

[4 marks]

(b) Use the boundary conditions: when $t = 0,\ x = 0$ and $\dfrac{dx}{dt} = 10$

$$x = e^{-0.3t}(A\cos 0.2t + B\sin 0.2t) + 2 \quad \Rightarrow\ 0 = A + 2 \ \Rightarrow\ A = -2$$

$$\frac{dx}{dt} = e^{-0.3t}(-0.2A\sin 0.2t + 0.2B\cos 0.2t)$$

$$- 0.3e^{-0.3t}(A\cos 0.2t + B\sin 0.2t)$$

$$\Rightarrow\ 10 = 0.2B - 0.3 \times A$$

$$10 = 0.2B + 0.6$$

$$B = 47$$

The particular solution is $x = e^{-0.3t}(-2\cos 0.2t + 47\sin 0.2t) + 2$

At maximum concentration, $\dfrac{dx}{dt} = 0$

$$e^{-0.3t}(0.4\sin 0.2t + 9.4\cos 0.2t) - 0.3e^{-0.3t}(-2\cos 0.2t + 47\sin 0.2t) = 0$$

$$4\sin 0.2t + 94\cos 0.2t + 6\cos 0.2t - 141\sin 0.2t = 0$$

$$-137\sin 0.2t + 100\cos 0.2t = 0$$

$$\tan 0.2t = \frac{100}{137}$$

$$0.2t = 0.631,\ \ldots$$

$$t = 3.15 \text{ weeks}$$

Concentration is then,

$$x = e^{-0.3\times 3.15}(-2\cos(0.2 \times 3.15) + 47\sin(0.2 \times 3.15) + 2$$

$$= 12.1\ \mu\text{g/ml}$$

[8 marks]

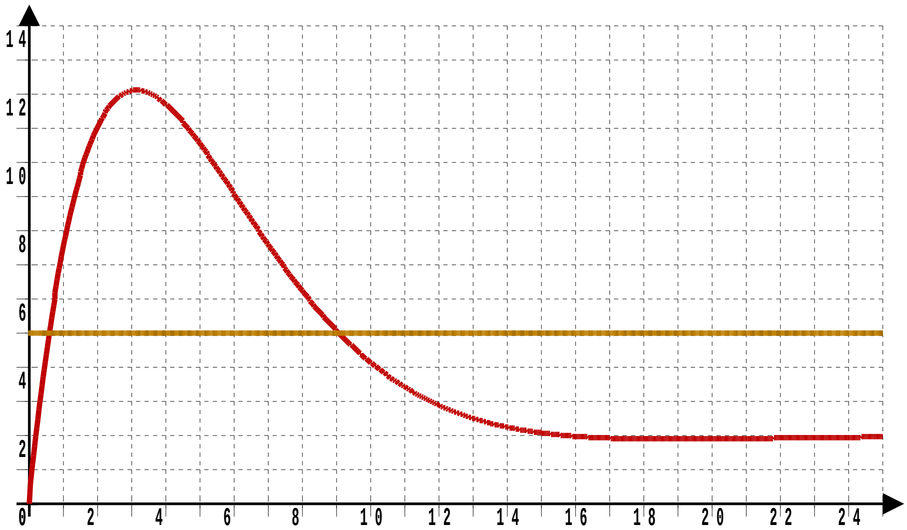
(c) Assume that any oscillations are negligible when $t = 10$

$$x = e^{-0.3 \times 10}(-2\cos(0.2 \times 10) + 47\sin(0.2 \times 10) + 2$$

$$= 4.17\,\mu\text{g/ml}$$

Which suggests it is safe to give the second dose after 10 weeks

A graph confirms the assumption about any oscillations being negligible after ten weeks.



[2 marks]

Answer 6

(i) The particle will exhibit (undamped) simple harmonic motion because

$\ddot{x} = -200\,x$ is of the form $\dfrac{d^2x}{dt^2} = -\omega^2 x$ with $\omega^2 = 200$

[1 mark]

(ii) $$\frac{d^2x}{dt^2} + 200x = 0$$

$m^2 + 200 = 0$ is the auxiliary equation

$$m^2 - \left(10\sqrt{2}\ \mathrm{i}\right)^2 = 0$$

$m = \pm 10\sqrt{2}\ \mathrm{i}$ (complex roots)

The general solution is $x = \left(A\,cos\left(10\sqrt{2}\ t\right) + B\,sin\left(10\sqrt{2}\ t\right)\right)$

Using the $t = 0$ boundary conditions; $x = 0.3$ and the velocity is 0 $\mathrm{m.s^{-1}}$

$x = A\,cos\,3t + B\,sin\,3t \quad \Rightarrow\ 0.3 = A$

$\dfrac{dx}{dt} = \left(-10\sqrt{2}\,A\,sin\left(10\sqrt{2}\ t\right) + 10\sqrt{2}\,B\,cos\left(10\sqrt{2}\ t\right)\right) \Rightarrow B = 0$

The particular solution is, $x = 0.3\,cos\left(10\sqrt{2}\ t\right)$

[7 marks]

(iii) Period : $$P = \frac{2\pi}{\omega}$$

$$= \frac{2\pi}{10\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\pi\sqrt{2}}{10} \text{ seconds}$$

Amplitude : 0.3 metres

[3 marks]

(iv) $$\frac{dx}{dt} = \left(-3\sqrt{2}\ sin\left(10\sqrt{2}\ t\right)\right)$$

$\therefore$ Maximum speed is $3\sqrt{2}$ $\mathrm{m.s^{-1}}$ (Approximately 4.24 $\mathrm{m.s^{-1}}$)

[2 marks]

Answer 7

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 2e^{-t}$$

(a) Obtain the complementary function by solving the homogeneous equation,

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0$$

$$m^2 + 5m + 6 = 0 \quad \text{is the auxiliary equation}$$

$$(m + 3)(m + 2) = 0$$

Either $m = -3$ or $m = -2$ (two distinct real roots)

$\therefore$ The complementary function is $x = Ae^{-3t} + Be^{-2t}$

For a particular integral try $x = Ue^{-t}$

in which case $\dfrac{dx}{dt} = -Ue^{-t}$

and $\dfrac{d^2x}{dt^2} = Ue^{-t}$

Substitute these into the question's differential equation...

$$Ue^{-t} - 5Ue^{-t} + 6Ue^{-t} = 2e^{-t}$$

$$2Ue^{-t} = 2e^{-t}$$

$$U = 1$$

$\therefore$ The particular integral is $x = e^{-t}$

The general solution is $x = Ae^{-3t} + Be^{-2t} + e^{-t}$

Use the boundary conditions: when $t = 0$, $x = 0$ and $\dfrac{dx}{dt} = 2$

$x = Ae^{-3t} + Be^{-2t} + e^{-t} \quad \Rightarrow \quad 0 = A + B + 1$

$\dfrac{dx}{dt} = -3Ae^{-3t} - 2Be^{-3t} - e^{-t} \quad \Rightarrow \quad 2 = -3A - 2B - 1$

Solve these simultaneously to get $A = -1$ and $B = 0$

The particular solution is $x = -e^{-3t} + e^{-t}$

[8 marks]

(b) At a maximum (or minimum) distance

$$\frac{dx}{dt} = 0$$

$$3e^{-3t} - e^{-t} = 0$$

Multiply through by e^{3t}

$$
\begin{aligned}
3 - e^{2t} &= 0 \\
e^{2t} &= 3 \\
2t &= ln\,3 \\
t &= \frac{1}{2}\,ln\,3 \\
&= ln\sqrt{3} \text{ seconds}
\end{aligned}
$$

At this time,

$$
\begin{aligned}
x &= -e^{-3\,ln\sqrt{3}} + e^{-\,ln\sqrt{3}} \\
&= -e^{ln\sqrt{3}^{\,-3}} + e^{ln\sqrt{3}^{\,-1}} \\
&= -\sqrt{3}^{\,-3} + \sqrt{3}^{\,-1} \\
&= -\frac{1}{3\sqrt{3}} + \frac{1}{\sqrt{3}} \\
&= \frac{-1+3}{3\sqrt{3}} \\
&= \frac{2}{3\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \\
&= \frac{2\sqrt{3}}{9} \qquad \square
\end{aligned}
$$

To determine the nature of the turning point...

$$
\begin{aligned}
\frac{d^2x}{dt^2} &= -9\,e^{-3t} + e^{-t} \\
&= -9\,e^{-3\,ln\sqrt{3}} + e^{-\,ln\sqrt{3}} \quad \text{using } t = ln\sqrt{3} \\
&= -9\,e^{ln\sqrt{3}^{\,-3}} + e^{ln\sqrt{3}^{\,-1}} \\
&= -9\sqrt{3}^{\,-3} + \sqrt{3}^{\,-1} \\
&= -\frac{9}{3\sqrt{3}} + \frac{1}{\sqrt{3}} \\
&= -\frac{2\sqrt{3}}{3}
\end{aligned}
$$

As this is negative, x is indeed a maximum.

[7 marks]

3.1 Coupled Systems

By definition, a coupled system comprises of two first order differential equations with two dependent variables and one independent variable. Typically, t is the independent variable and represents time. The dependent variables, x and y, each depend of t. In biology x and y may represent the populations of two species of animal that have a symbiotic relationship. For example, one may be a predator and the other, prey. The population of each depends both on how large their own population is, and the size of the other's.

Coupled System Solution Technique

Coupled first-order linear differential equations may be solved by eliminating one of the dependent variables to form a second-order differential equation. This is then solved using methods previously studied.

3.2 Example

(i) Find the general solutions to the differential equations,

$$\frac{dx}{dt} = x + 5y \qquad\qquad \frac{dy}{dt} = -x - 3y$$

Work on first equation: $\frac{dx}{dt} = x + 5y$

$$5y = \frac{dx}{dt} - x \qquad \text{Making } y \text{ the subject}$$

$$y = 0.2\frac{dx}{dt} - 0.2x$$

$$\frac{dy}{dt} = 0.2\frac{d^2x}{dt^2} - 0.2\frac{dx}{dt} \qquad \text{By differentiation}$$

Work on second equation: $\frac{dy}{dt} = -x - 3y$

$$0.2\frac{d^2x}{dt^2} - 0.2\frac{dx}{dt} = -x - 3\left(0.2\frac{dx}{dt} - 0.2x\right)$$

$$= -x - 0.6\frac{dx}{dt} + 0.6x$$

$$0.2\frac{d^2x}{dt^2} + 0.4\frac{dx}{dt} + 0.4 = 0$$

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2 = 0$$

Next, write down and solve the auxiliary equation...

$$m^2 + 2m + 2 = 0$$

$$m = \frac{-2 \pm \sqrt{4 - 4 \times 1 \times 2}}{2}$$

$$= \frac{-2 \pm \sqrt{(-1)(4)}}{2}$$

$$= -1 \pm \mathrm{i}$$

$$x = e^{-t}(A\cos t + B\sin t)$$

Differentiate this as we need a "$y =$" companion to this "$x =$"

$$\frac{dx}{dt} = e^{-t}(-A\sin t + B\cos t) - e^{-t}(A\cos t + B\sin t)$$

$$= e^{-t}((B - A)\cos t + (-A - B)\sin t)$$

Go back to the first equation: $\dfrac{dx}{dt} = x + 5y$

$$e^{-t}((B - A)\cos t + (-A - B)\sin t) = e^{-t}(A\cos t + B\sin t) + 5y$$

$$5y = e^{-t}((B - 2A)\cos t - (A + 2B)\sin t)$$

$$y = \frac{1}{5}e^{-t}((B - 2A)\cos t - (A + 2B)\sin t)$$

The general solutions are,

$$x = e^{-t}(A\cos t + B\sin t)$$

$$y = \frac{1}{5}e^{-t}((B - 2A)\cos t - (A + 2B)\sin t)$$

[8 marks]

(ii) Given that at time $t = 0$, $x = 1$ and $y = 2$, find the particular solutions

$$x = e^{-t}(A\cos t + B\sin t)$$

$$1 = A$$

$$y = \frac{1}{5}e^{-t}((B - 2A)\cos t - (A + 2B)\sin t)$$

$$2 = \frac{1}{5}(B - 2A)$$

$$10 = B - 2$$

$$B = 12$$

The particular solutions are,

$$x = e^{-t}(\cos t + 12\sin t)$$

$$y = e^{-t}(2\cos t - 5\sin t)$$

[4 marks]

3.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 32

Question 1

(i) Find the general solutions to the differential equations,

$$\frac{dx}{dt} = -x + 6y \qquad\qquad \frac{dy}{dt} = x - 2y$$

[8 marks]

(ii) Given that at time $t = 0$, $x = 2$ and $y = 0$, find the particular solutions

[4 marks]

Question 2

A tank of water on O'Kevin's farm contains two different types of chemical that react with each other. The rates of change of each chemical can be modelled using the following differential equations;

Photograph by Tracy's Photography

$$\frac{dx}{dt} = -3x + 2y$$
$$\frac{dy}{dt} = -2x + y$$

In the equations, x is the number of litres of chemical X and y is the number of litres of chemical Y at time t hours.

Initially there is 100 litres of chemical X and 200 litres of chemical Y in the tank.

(i) Show that the solutions to the differential equations can be written as, $x = P\,e^{-t}$ and $y = Q\,e^{-t}$ where P and Q are functions of t to be found.

[8 marks]

(ii) Find, correct to three significant figures, the amount of each chemical in O'Kevin's tank at $t = 2$ hours.

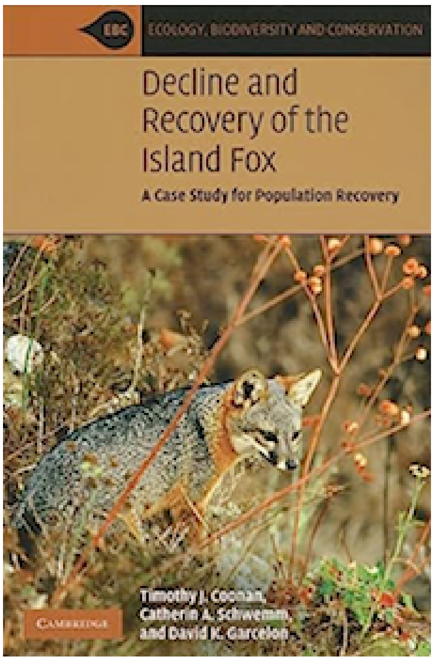
[2 marks]

(iii) Use the model to describe what happens to the amount of each chemical in O'Kevin's tank as t gets large.

[2 marks]

Question 3

At the start of 2023, Professor Oscar Workfast, and his rival Professor Boris Beach initiated an annual survey on the number of Weasels, x, and Island Foxes, y, on one of the Channel Islands of California. The Weasels preys on the Island Fox. The mathematical model they are developing involves the following two differential equations;



$$\frac{dx}{dt} = 0.2x + 0.2y$$

$$\frac{dy}{dt} = -0.5x + 0.4y$$

(i) Show that $\dfrac{d^2x}{dt^2} - 0.6\dfrac{dx}{dt} + 0.18x = 0$

[3 marks]

(ii) Prof Workfast claims that the general solution for the number of weasels at time t is $x = e^{\alpha t}\,(A\cos\beta t + B\sin\beta t)$.
Show that this is correct and find the values of the constants α and β.

[4 marks]

(iii) Not to be outdone, Prof BB claims that the number of Island Foxes will be an equation of the form $y = P\,e^{\alpha t}\,(Q\cos\beta t + R\sin\beta t)$.
Show that this is also correct and find the value of the constant P, and determine the functions Q and R in terms of A and B.

[3 marks]

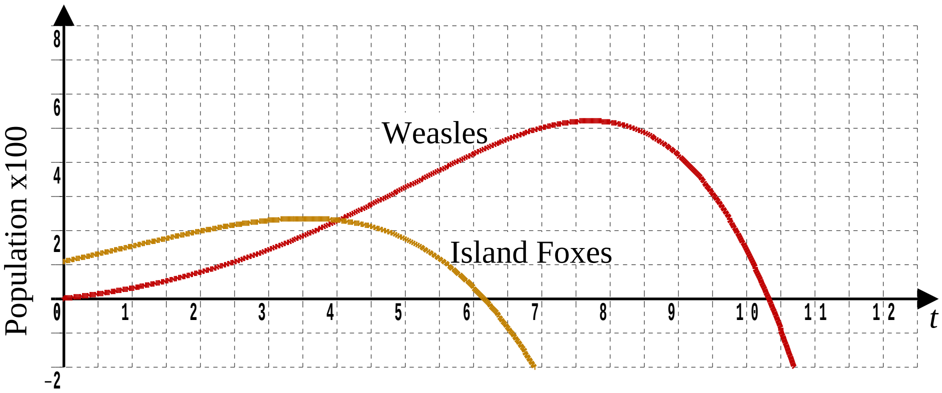
(iv) Prof Beach loves Island Foxes and despises the Weasels that prey upon them. Prof Workfast worships the Weasels and views the Island Foxes as nothing more than food for his Weasels. In the survey, the professors count 3 Weasels and 111 Island Foxes.
During which year does the model predict the Island Foxes die out ?

[5 marks]

(v) How many Weasels will there be when the Island Foxes die out ?

[1 mark]

(vi) The particular solutions to the model are graphed below.



With reference to the graph, comment on the model.

[2 marks]

3.4 Answers to 3.5 Exercise

Answer 1

(i) $\dfrac{dx}{dt} = -x + 6y \qquad \dfrac{dy}{dt} = x - 2y$

Work on first equation: $\dfrac{dx}{dt} = -x + 6y$

$$y = \frac{1}{6}\frac{dx}{dt} + \frac{1}{6}x \qquad \text{Making } y \text{ the subject}$$

$$\frac{dy}{dt} = \frac{1}{6}\frac{d^2x}{dt^2} + \frac{1}{6}\frac{dx}{dt} \qquad \text{By differentiation}$$

Work on second equation: $\dfrac{dy}{dt} = x - 2y$

$$\frac{1}{6}\frac{d^2x}{dt^2} + \frac{1}{6}\frac{dx}{dt} = x - 2\left(\frac{1}{6}\frac{dx}{dt} + \frac{1}{6}x\right)$$

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} = 6x - 2\frac{dx}{dt} - 2x$$

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} - 4x = 0$$

$$m^2 + 3m - 4 = 0$$

$$(m - 1)(m + 4) = 0$$

Either $m = 1$ or $m = -4$ (two distinct real roots)

$$x = Ae^t + Be^{-4t}$$

Differentiate this as we need a "$y =$" companion to this "$x =$"

$$\frac{dx}{dt} = Ae^t - 4Be^{-4t}$$

Go back to the first equation: $\dfrac{dx}{dt} = -x + 6y$

$$Ae^t - 4Be^{-4t} = -Ae^t - Be^{-4t} + 6y$$

$$6y = 2Ae^t - 3Be^{-4t}$$

$$y = \frac{1}{3}Ae^t - \frac{1}{2}Be^{-4t}$$

General Solutions: $x = Ae^t + Be^{-4t}$

$$y = \frac{1}{3}Ae^t - \frac{1}{2}Be^{-4t}$$

[8 marks]

(ii) Boundary conditions; $t = 0$, $x = 2$ and $y = 0$

$$x = Ae^{t} + Be^{-4t} \qquad \Rightarrow \; 2 = A + B$$

$$y = \frac{1}{3}Ae^{t} - \frac{1}{2}Be^{-4t} \qquad \Rightarrow \; 0 = \frac{1}{3}A - \frac{1}{2}B$$

Simultaneous equations,

$$2A + 2B = 4$$

SUBTRACT $\quad 2A - 3B = 0$

$$5B = 4$$

$$B = \frac{4}{5} \text{ and } A = \frac{6}{5}$$

The particular solutions are,

$$x = \frac{6}{5}e^{t} + \frac{4}{5}e^{-4t}$$

$$y = \frac{2}{5}e^{t} - \frac{2}{5}e^{-4t}$$

[4 marks]

Answer 2

(i) $\dfrac{dx}{dt} = -3x + 2y \qquad \dfrac{dy}{dt} = -2x + y$

Work on first equation: $\dfrac{dx}{dt} = -3x + 2y$

$$y = \frac{1}{2}\frac{dx}{dt} + \frac{3}{2}x \qquad \text{Making } y \text{ the subject}$$

$$\frac{dy}{dt} = \frac{1}{2}\frac{d^2x}{dt^2} + \frac{3}{2}\frac{dx}{dt} \qquad \text{By differentiation}$$

Work on second equation: $\dfrac{dy}{dt} = -2x + y$

$$\frac{1}{2}\frac{d^2x}{dt^2} + \frac{3}{2}\frac{dx}{dt} = -2x + \frac{1}{2}\frac{dx}{dt} + \frac{3}{2}x$$

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} = -4x + \frac{dx}{dt} + 3x$$

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = 0$$

$$m^2 + 2m + 1 = 0$$

$$(m + 1)^2 = 0$$

$$m = -1 \qquad \text{(repeated root)}$$

$$x = (A + Bt)\,e^{-t}$$

Differentiate this as we need a "$y =$" companion to this "$x =$"

$$\frac{dx}{dt} = (-1)(A + Bt)e^{-t} + Be^{-t}$$

$$= (B - A - Bt)e^{-t}$$

Go back to the first equation: $\dfrac{dx}{dt} = -3x + 2y$

$$(B - A - Bt)e^{-t} = -3(A + Bt)e^{-t} + 2y$$

$$2y = (B - A - Bt + 3A + 3Bt)e^{-t}$$

$$y = \frac{1}{2}(B + 2A + 2Bt)e^{-t}$$

General Solutions: $x = (A + Bt)e^{-t}$

$$y = \frac{1}{2}(B + 2A + 2Bt)e^{-t}$$

Boundary conditions; $t = 0$, $x = 100$ and $y = 200$

$x = (A + Bt)e^{-t} \quad \Rightarrow \quad 100 = A$

$y = \dfrac{1}{2}(B + 2A + 2Bt)e^{-t} \quad \Rightarrow \quad 200 = \dfrac{1}{2}B + A$ so $B = 200$

The particular solutions are,

$$x = 100(1 + 2t)e^{-t}$$

$$y = 200(1 + t)e^{-t}$$

[8 marks]

(ii) $x = 100(1 + 2 \times 2)e^{-2} = 67.7$ litres

$y = 200(1 + 2)e^{-2} = 81.2$ litres

[2 marks]

(iii) As t becomes large, e^{-t} tends towards zero.

So the amount of both chemicals will tend towards zero.

[2 marks]

Answer 3

(i) $$\frac{dx}{dt} = 0.2\,x + 0.2\,y \qquad \frac{dy}{dt} = -0.5\,x + 0.4\,y$$

Work on first equation: $\frac{dx}{dt} = 0.2\,x + 0.2\,y$

$y = 5\frac{dx}{dt} - x$ Making y the subject

$\frac{dy}{dt} = 5\frac{d^2x}{dt^2} - \frac{dx}{dt}$ By differentiation

Work on second equation: $\frac{dy}{dt} = -0.5\,x + 0.4\,y$

$$5\frac{d^2x}{dt^2} - \frac{dx}{dt} = -0.5\,x + 0.4\left(5\frac{dx}{dt} - x\right)$$

$$5\frac{d^2x}{dt^2} - \frac{dx}{dt} = -0.5\,x + 2\frac{dx}{dt} - 0.4\,x$$

$$5\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 0.9x = 0$$

$$\frac{d^2x}{dt^2} - 0.6\frac{dx}{dt} + 0.18x = 0$$

[3 marks]

(ii) The auxiliary equation is,

$$m^2 - 0.6m + 0.18 = 0$$

$$100\,m^2 - 60m + 18 = 0$$

$$50\,m^2 - 30m + 9 = 0$$

$$m = \frac{30 \pm \sqrt{900 - 4 \times 50 \times 9}}{2 \times 50}$$

$$= \frac{30 \pm \sqrt{(-1)(900)}}{100}$$

$$= 0.3 \pm 0.3\mathrm{i}$$

General solution : $x = e^{0.3\,t}(\,A\,cos\,(0.3t) + B\,sin\,(0.3t))$

That is, $\alpha = \beta = 0.3$

[4 marks]

(iii) Differentiate part (ii) answer as we need a "y =" companion to this "x ="

$$\frac{dx}{dt} = e^{0.3t}(-0.3A\,sin(0.3t) + 0.3B\,cos(0.3t)\,)$$

$$+ 0.3e^{0.3t}(\,A\,cos(0.3t) + B\,sin(0.3t))$$

$$= 0.3\,e^{0.3t}(\,(\,B + A\,)\,cos(0.3t) + (\,B - A\,)\,sin(0.3t)\,)$$

Go back to the first equation: $\frac{dx}{dt} = 0.2\,x + 0.2\,y$

$$y = 5\frac{dx}{dt} - x$$

$$y = 1.5\,e^{0.3t}((B + A)\,cos(0.3t) + (B - A)\,sin(0.3t)) - \left(e^{0.3t}(Acos(0.3t) + Bsin(0.3t))\right)$$

$$y = e^{0.3t}\{(1.5B + 1.5A - A)\,cos(0.3t) + (1.5B - 1.5A - B)\,sin(0.3t)\}$$

$$y = e^{0.3t}\{(\,0.5A + 1.5B)\,cos(0.3t) + (-1.5A + 0.5B\,)\,sin(0.3t)\}$$

$$y = 0.1\,e^{0.3t}\{(\,5A + 15B)\,cos(0.3t) + (-15A + 5B\,)\,sin(0.3t)\}$$

[3 marks]

(iv) Use the t = 0 conditions, x = 3 (Weasels), y = 111 (Foxes)

$$x = e^{0.3t}(\,A\,cos(0.3t) + B\,sin(0.3t)) \Rightarrow A = 3$$

$$y = 0.1\,e^{0.3t}\{(\,5A + 15B)\,cos(0.3t) + (-15A + 5B\,)\,sin(0.3t)\}$$

$$\Rightarrow 111 = 0.1\,(\,5A + 15B\,)$$

$$1110 = 15 + 15B$$

$$B = 73$$

Particular solutions: $x = e^{0.3t}(\,3\,cos(0.3t) + 73\,sin(0.3t))$

$$y = 0.1\,e^{0.3t}\{\,1110\,cos(0.3t) + 320\,sin(0.3t)\}$$

To find the time when $y = 0$?

$$1110\,cos(0.3t) + 320\,sin(0.3t) = 0$$

$$320\,sin(0.3t) = -1110\,cos(0.3t)$$

$$tan(0.3t) = -\frac{1110}{320}$$

$$= -3.469$$

$$0.3t = -1.290,\ 1.851,\ ...$$

$$t = 6.172 \text{ years}$$

Year Islands Foxes die out is $2023 + 6.172 =$ During 2029

[5 marks]

(v) $x = e^{0.3t}(3\,cos(0.3t) + 73\,sin(0.3t))$

The number of Weasels when $t = 6.172$ will be...

$x = e^{0.3 \times 6.172}(3\,cos(0.3 \times 6.172) + 73\,sin(0.3 \times 6.172))$

$= 441.5$

[1 mark]

(vi) The graph shows that the Island Foxes cannot be the only source of food for the Weasels, as their population goes on rising after the Island Foxes have died out. The question seemed to suggest that the populations of the two species were interrelated, but if this were so, the Island Foxes demise would have also caused the population of Weasels to suffer. This model is, perhaps, a bit too simple to realistically capture population dynamics.

[2 marks]