Push The Pace #1
Answers

Further A-Level Pure Mathematics
Push The Pace Revision Papers

Answer 1

For the cylinder formed by rotating the stem of the funnel about the y-axis,

$$V_{CYLINDER} = \pi \times 1^2 \times 4$$
$$= 4\pi$$

For the bowl formed by rotating the arc of the circle about the y-axis,

$$V_{BOWL} = \pi \int_4^9 37 - (y - 10)^2 \, dy$$
$$= \pi \int_4^9 37 - y^2 + 20y - 100 \, dy$$
$$= \pi \int_4^9 -y^2 + 20y - 63 \, dy$$
$$= \pi \left[-\frac{y^3}{3} + 10y^2 - 63y \right]_4^9$$
$$= \pi \left[-243 + 810 - 567 + \frac{64}{3} - 160 + 252 \right]$$
$$= \frac{340\pi}{3}$$

Total Volume is $\dfrac{352\pi}{3}$ cm^3 (About 368.6 cm^3)

[3 marks]

Answer 2

$$\mathbf{AB} = k\mathbf{I}$$

$$\begin{pmatrix} x+1 & 2 \\ x+2 & -3 \end{pmatrix}\begin{pmatrix} x-4 & x-2 \\ 0 & -2 \end{pmatrix} = k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} (x+1)(x-4) & x^2 - x - 6 \\ (x+2)(x-4) & x^2 + 2 \end{pmatrix} = k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} (x+1)(x-4) & (x+2)(x-3) \\ (x+2)(x-4) & x^2 + 2 \end{pmatrix} = k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus, $x = -2$ (to get the zeros) and the multiplication is then,

$$\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} = k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So $k = 6$ when $x = -2$

[3 marks]

Answer 3

$$\sum_{r=1}^{k} (3r - k) = 90$$

$$3 \sum_{r=1}^{k} r - k \sum_{r=1}^{k} 1 = 90$$

$$3 \times \frac{k(k+1)}{2} - k^2 = 90$$

$$3k(k+1) - 2k^2 = 180$$

$$k^2 + 3k - 180 = 0$$

$$(k+15)(k-12) = 0$$

Either $k = -15$ (reject) or $k = 12$

The only answer is $k = 12$

[3 marks]

Answer 4

(a) $$4x^2 + 10x - 24 = 4\left[x^2 + \frac{5}{2}x \right] - 24$$

$$= 4\left[\left(x + \frac{5}{4} \right)^2 - \frac{25}{16} \right] - 24$$

$$= 4\left(x + \frac{5}{4} \right)^2 - \frac{25}{4} - 24$$

$$= 4\left(x + \frac{5}{4} \right)^2 - \frac{121}{4} \qquad \left(a = 4,\ b = \frac{5}{4},\ c = -\frac{121}{4} \right)$$

[3 marks]

(b) $$= 4\left[\left(x + \frac{5}{4} \right)^2 - \frac{121}{16} \right]$$

$$= 4\left[\left(x + \frac{5}{4} \right)^2 - \left(\frac{11}{4} \right)^2 \right]$$

Now to use the formulae booklet (page 12) where we are told that,

$$\int \frac{1}{\sqrt{x^2 - a^2}}\, dx = arcosh\left(\frac{x}{a} \right),\ ln\left\{ x + \sqrt{x^2 - a^2} \right\} + c \quad \text{for } x > a$$

By all means use the first result, we're going for the second...

$$\int_3^5 \frac{6}{\sqrt{4x^2 + 10x - 24}}\, dx$$

$$= \frac{1}{2}\int_3^5 \frac{6}{\sqrt{\left(x + \frac{5}{4} \right)^2 - \left(\frac{11}{4} \right)^2}}\, dx$$

$$= 3\left[ln\left\{ x + \frac{5}{4} + \sqrt{\left(x + \frac{5}{4} \right)^2 - \left(\frac{11}{4} \right)^2} \right\} \right]_3^5$$

$$= 3\left[ln\left\{ \frac{25}{4} + \sqrt{\left(\frac{25}{4} \right)^2 - \left(\frac{11}{4} \right)^2} \right\} - ln\left\{ \frac{17}{4} + \sqrt{\left(\frac{17}{4} \right)^2 - \left(\frac{11}{4} \right)^2} \right\} \right]$$

$$= 3\left[ln\left\{ \frac{25}{4} + \frac{6\sqrt{14}}{4} \right\} - ln\left\{ \frac{17}{4} + \frac{2\sqrt{42}}{4} \right\} \right]$$

$$= 3\,ln\left\{ \frac{25 + 6\sqrt{14}}{17 + 2\sqrt{42}} \right\}$$

$$= 1.379$$

[5 marks]

Answer 5

Begin by solving the homogeneous equation, $3\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 2y = 0$

This has auxiliary equation, $3m^2 + 5m - 2 = 0$

$$(3m - 1)(m + 2) = 0$$

Either $m = \frac{1}{3}$ or $m = -2$ (Two distinct real roots)

The general solution is $y = Ae^{\frac{1}{3}x} + Be^{-2x}$

For the particular integral try $y = Ux^2 + Vx + W$

in which case $\frac{dy}{dx} = 2Ux + V$ and $\frac{d^2y}{dx^2} = 2U$

Substitute these into the question's differential equation...

$$3 \times 2U + 5(2Ux + V) - 2\left(Ux^2 + Vx + W\right) = 8 + 6x - 2x^2$$

$$6U + 10Ux + 5V - 2Ux^2 - 2Vx - 2W = 8 + 6x - 2x^2$$

$$(6U + 5V - 2W) + (10U - 2V)x - 2Ux^2 = 8 + 6x - 2x^2$$

Match up coefficients of x^2 : $U = 1$

Match up coefficients of x : $V = 2$

Terms independent of x : $6 + 10 - 2W = 8$ gives $W = 4$

$\therefore$ The particular integral is $y = x^2 + 2x + 4$

The general solution is $y = Ae^{\frac{1}{3}x} + Be^{-2x} + x^2 + 2x + 4$

from which $\frac{dy}{dx} = \frac{A}{3}e^{\frac{1}{3}} - 2Be^{-2} + 2x + 2$

From the boundary conditions, $y = 6$ and $\frac{dy}{dx} = 5$ when $x = 0$

$$6 = A + B + 4 \text{ and } 5 = \frac{A}{3} - 2B + 2$$

Solving these simultaneously...

$$A + B = 2$$

SUBTRACT $A - 6B = 9$

$7B = -7$ so $B = -1$ and $A = 3$

The particular solution is $y = 3e^{\frac{1}{3}x} - e^{-2x} + x^2 + 2x + 4$

[12 marks]

Answer 6
You could use the various rules for sum or products of roots.
Here is a "first principles" approach.

$$4x^3 - 12x^2 - 13x + k = 0$$

Divide through by 4

$$x^3 - 3x^2 - \frac{13}{4}x + \frac{k}{4} = 0$$

Let the roots be $(a - d)$, a, $(a + d)$ which are in arithmetic progression.

(For some constant a)

$$\begin{aligned}
\text{In consequence, } (x - a + d)(x - a)(x - a - d) &= 0 \\
\left((x - a)^2 - d^2\right)\left(x - a\right) &= 0 \\
\left(x^2 - 2ax + a^2 - d^2\right)\left(x - a\right) &= 0 \\
x^3 - 2ax^2 + \left(a^2 - d^2\right)x & \\
- ax^2 + 2a^2x - a^3 + ad^2 &= 0 \\
x^3 - 3ax^2 + \left(3a^2 - d^2\right)x - a^3 + ad^2 &= 0
\end{aligned}$$

Match up coefficients of x^2 : $a = 1$

Match up coefficients of x : $3a^2 - d^2 = -\frac{13}{4}$ so $d^2 = 3 + \frac{13}{4}$

$$d = \pm\frac{5}{2}$$

So the roots are, $x = -\frac{3}{2}$, $x = 1$, and $x = \frac{7}{2}$

[6 marks]

Answer 7

(a) In matrix form the equations are, $\begin{pmatrix} 1 & 2 & 0 \\ 2 & -5 & 3 \\ 0 & 6 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \\ 0 \end{pmatrix}$

However, the 3×3 matrix has a determinant of zero.
To see this either use your calculator or, expand along the upper row,

$$\begin{aligned}
\begin{vmatrix} 1 & 2 & 0 \\ 2 & -5 & 3 \\ 0 & 6 & -2 \end{vmatrix} &= (1)\begin{vmatrix} -5 & 3 \\ 6 & -2 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ 0 & -2 \end{vmatrix} + 0\begin{vmatrix} 2 & -5 \\ 0 & 6 \end{vmatrix} \\
&= 10 - 18 + 8 \\
&= 0 \quad \text{as claimed.}
\end{aligned}$$

This means that there is no unique solutions and so the planes do not meet at a single point.

[5 marks]

(b) Look at the normal vectors which are $\begin{pmatrix}1\\2\\0\end{pmatrix}$, $\begin{pmatrix}2\\-5\\6\end{pmatrix}$ and $\begin{pmatrix}0\\3\\2\end{pmatrix}$ but none of these are parallel.

Multiply the upper row by 15, the middle by 6 and the lower by 5

$$\begin{pmatrix}15 & 30 & 0\\12 & -30 & 18\\0 & 30 & -10\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}45\\48\\0\end{pmatrix}$$

Combine upper and middle equations,

$$\begin{array}{rrcrcrcl} & 15x & + & 30y & + & 0z & = & 45\\ \text{ADD} & 12x & - & 30y & + & 18z & = & 48\\ \hline & & & 27x & + & 18z & = & 93\\ & & & 9x & + & 6z & = & 31 \quad * \end{array}$$

Combine lower and middle equations,

$$\begin{array}{rrcrcrcl} & 0x & + & 30y & - & 10z & = & 0\\ \text{ADD} & 12x & - & 30y & + & 18z & = & 48\\ \hline & & & 12x & + & 8z & = & 48\\ & & & 3x & + & 2z & = & 12 \quad * \end{array}$$

As the two starred equations are not consistent, the three planes form a prism.

[6 BONUS marks]