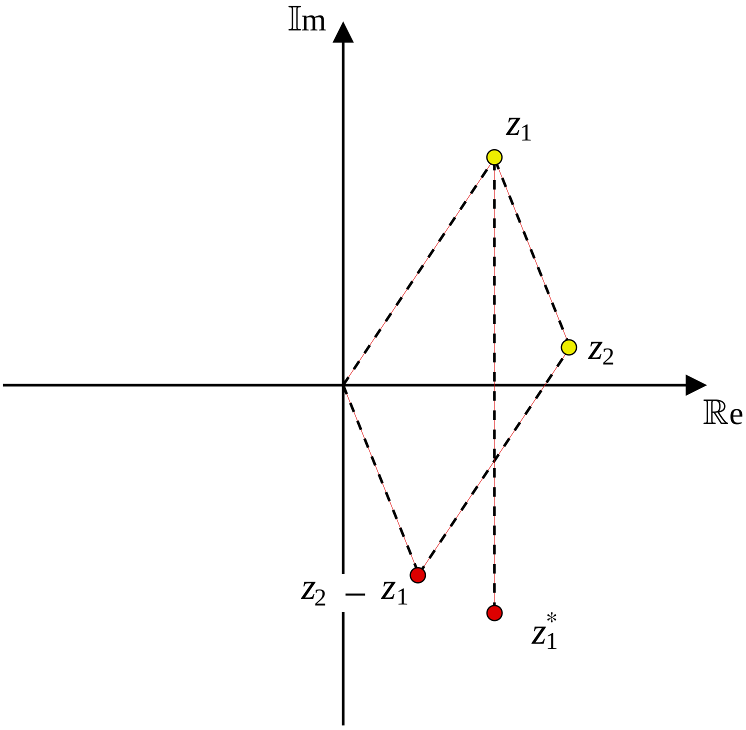
Push The Pace #2
Answers

Further A-Level Pure Mathematics
Push The Pace Revision Papers

Answer 1
(a)



[2 marks]

(b)

$$\frac{z_2 - z_1}{z_1^*} = \frac{3 + i - 1 - 2i}{1 - 2i}$$

$$= \frac{2 - i}{1 - 2i} \times \frac{1 + 2i}{1 + 2i}$$

$$= \frac{2 - i + 4i - 2i^2}{1 - 4i^2}$$

$$= \frac{4 + 3i}{5}$$

$$= \frac{4}{5} + \frac{3}{5}i \qquad \left(a = \frac{4}{5}, b = \frac{3}{5}\right)$$

[2 marks]

Answer 2

(a) $f(x) = cosh^3 x - 3\,cosh\,x$

$$f'(x) = 3\,cosh^2 x\,sinh\,x - 3\,sinh\,x$$
$$= 3\,sinh\,x\left(cosh^2 x - 1\right)$$
$$= 3\,sinh\,x\left(sinh^2 x\right) \qquad \text{using the identity } cosh^2 x - sinh^2 x = 1$$
$$= 3\,sinh^3 x$$

For a stationary point, $f'(x) = 0$

$$3\,sinh^3 x = 0$$
$$sinh\,x = 0$$
$$x = 0$$

So the graph of $f(x)$ only has one stationary point which is at the origin.

[4 marks]

(b) $f''(x) = 9\,sinh^2 x\,cosh\,x$

but $f''(0) = 0$ so this method has failed !

Instead look at the gradient either side of $x = 0$

There is no need to worry about how close to go to $x = 0$ as we know that there are no other stationary points.

$f'(-1) = 3\,sinh^3(-1) = -4.86$, negative gradient

$f'(+1) = 3\,sinh^3(+1) = +4.87$, positive gradient

The stationary point is a minimum.

[3 marks]

(c) We know the graph has a minimum at when $x = 0$

This has a value of $f(0) = cosh^3(0) - 3\,cosh(0)$

$$= -2$$

So the greatest range is $-2 \leqslant f(x) < \infty$

[1 mark]

Answer 3

We are told that $z^4 = 9 - 3\sqrt{3}\,\mathrm{i}$

Complex variable equations of the form $z^n = c$ have roots that lie on a circle with centre at the origin so we just need the radius of this circle (because we don't care where on the circle the roots are; we don't need to worry about angles).

$$\begin{aligned} \left|z^4\right| &= \left|9 - 3\sqrt{3}\,\mathrm{i}\right| \\ &= \sqrt{108} \\ \therefore z^4 &= \sqrt{108}\ \mathit{arg}\ \text{(angle we don't care about)} \\ z &= 108^{\frac{1}{8}}\ \mathit{arg}\ \text{(angles we don't care about)} \end{aligned}$$

$$\text{Equation of circle is: } x^2 + y^2 = 108^{\frac{1}{4}}$$

$$x^2 + y^2 = 3.22...$$

[4 marks]

Answer 4

(a) Area $= \dfrac{1}{2}\int_0^{\pi} r^2\, d\theta$

$$= \frac{1}{2}\int_0^{\pi} \sin\theta\, e^{\frac{2}{3}\cos\theta}\, d\theta$$

$$= \left(-\frac{3}{4}\right)\int_0^{\pi}\left(-\frac{2}{3}\sin\theta\right) e^{\frac{2}{3}\cos\theta}\, d\theta \text{ setting up chain rule backwards}$$

$$= \left(-\frac{3}{4}\right)\left[\, e^{\frac{2}{3}\cos\theta}\,\right]_0^{\pi}$$

$$= \left(-\frac{3}{4}\right)\left[\, e^{-\frac{2}{3}} - e^{\frac{2}{3}}\,\right]$$

$$= \frac{3}{4}\left(e^{\frac{2}{3}} - e^{-\frac{2}{3}}\right)$$

[4 marks]

(b) Greatest r occurs when $\dfrac{dr}{d\theta} = 0$

$$\frac{d}{d\theta}\left(\sqrt{\sin\theta}\; e^{\frac{1}{3}\cos\theta}\right) = 0$$

$$\sqrt{\sin\theta}\left(-\frac{1}{3}\sin\theta\right)e^{\frac{1}{3}\cos\theta} + \frac{1}{2}\,(\sin\theta)^{-\frac{1}{2}}\cos\theta\, e^{\frac{1}{3}\cos\theta} = 0$$

$$\frac{1}{6}\left(e^{\frac{1}{3}\cos\theta}\right)\left(-2\,(\sin\theta)^{\frac{3}{2}} + 3(\sin\theta)^{-\frac{1}{2}}\cos\theta\right) = 0$$

$$\left(-2\,(\sin\theta)^{\frac{3}{2}} + 3(\sin\theta)^{-\frac{1}{2}}\cos\theta\right) = 0$$

$$-2\sin^2\theta + 3\cos\theta = 0$$

$$-2\left(1 - \cos^2\theta\right) + 3\cos\theta = 0$$

$$2\cos^2\theta + 3\cos\theta - 2 = 0$$

$$(2\cos\theta - 1)(\cos\theta + 2) = 0$$

Either $\cos\theta = \dfrac{1}{2}$ or $\cos\theta = -2$ (no solutions)

Because of the $\sqrt{\sin\theta}$, $\sin\theta \geqslant 0$

so, from $\cos\theta = \dfrac{1}{2}$ we get $\sin\theta = \dfrac{\sqrt{3}}{2}$

Greatest value of $r = \sqrt{\sin\theta}\; e^{\frac{1}{3}\cos\theta}$

$$r_{MAX} = \sqrt{\frac{\sqrt{3}}{2}}\, e^{\frac{1}{6}}$$

[7 marks]

Answer 5

The Maclaurin Series

A given function, $f(x)$, may be written as the polynomial,

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \ldots + \frac{f^{(r)}(0)}{r!}\,x^r + \ldots$$

provided that $f(0)$, $f'(0)$, $f''(0)$, ..., $f^{(r)}(0)$, ... all have finite values.

(a) Let $f(x) = ln\left(\frac{1}{2} + cos\,x\right)$ $\quad : \quad f(0) = ln\left(\frac{3}{2}\right)$

$$f'(x) = \frac{-\ sin\,x}{\frac{1}{2} + cos\,x} = \frac{-2\,sin\,x}{1 + 2\,cos\,x} \quad : \quad f'(0) = 0$$

$$f''(x) = \frac{(1 + 2\,cos\,x)(-2\,cos\,x) - (-2\,sin\,x)(-2\,sin\,x)}{(1 + 2\,cos\,x)^2}$$

$$= \frac{-2\,cos\,x - 4\,cos^2 x - 4\,sin^2 x}{(1 + 2\,cos\,x)^2}$$

$$= \frac{-2\,cos\,x - 4}{(1 + 2\,cos\,x)^2} \quad : \quad f''(0) = \frac{-6}{9} = -\frac{2}{3}$$

$$\therefore\ ln\left(\frac{1}{2} + cos\,x\right) = ln\left(\frac{3}{2}\right) - \frac{1}{3}\,x^2 + \ldots$$

[4 marks]

(b)

$$ln\left(\frac{1}{2} + cos\,x\right) = 0$$

$$\frac{1}{2} + cos\,x = 1$$

$$cos\,x = \frac{1}{2}$$

$$x = \frac{\pi}{3}$$

So we have that, $ln\left(\frac{3}{2}\right) - \frac{1}{3}\left(\frac{\pi}{3}\right)^2 + \ldots = 0$

$$\frac{\pi^2}{27} \approx ln\left(\frac{3}{2}\right)$$

$$\pi \approx \sqrt{27\,ln\left(\frac{3}{2}\right)}$$

$$\pi \approx 3\sqrt{3\,ln\left(\frac{3}{2}\right)}$$

[3 marks]

Answer 6

(a) In the working the limits are omitted to reduce clutter...

$$
\begin{aligned}
I &= \int e^{2t} \sin t \, dt \\
&= e^{2t} \, (-\cos t) - \int 2 \, e^{2t} (-\cos t) \, dt \\
&= -e^{2t} \cos t + 2 \int e^{2t} \cos t \, dt \\
&= -e^{2t} \cos t + 2 \, e^{2t} \sin t - 2 \int 2 \, e^{2t} \sin t \, dt \\
I &= -e^{2t} \cos t + 2 \, e^{2t} \sin t - 4 \, I \\
5 \, I &= -e^{2t} \cos t + 2 \, e^{2t} \sin t \\
I &= \left[\frac{2}{5} e^{2t} \sin t - \frac{1}{5} e^{2t} \cos t \right]_a^b
\end{aligned}
$$

[6 marks]

(b) The question is requiring that we solve the differential equation.

$$\frac{dv}{dt} + v = 5e^t \sin t$$

The equation can be solved using the integrating factor method $\left(\text{with } P(t) = 1 \text{ and } Q(t) = 5e^t \sin t\right)$

$$\int P(t)\,dt = \int 1\,dt = t$$

If there are no occurrences of the modulus function the constant of integration may be omitted, as done here.

$$I_{FACTOR} = e^{\int P(x)\,dx} = e^t$$

$$e^t\frac{dv}{dt} + e^t v = 5e^{2t}\sin t$$

integrate both sides with respect to t

$$\int e^t\frac{dv}{dt} + e^t v\,dt = 5\int e^{2t}\sin t\,dt$$

$$e^t v = 5\left[\frac{2}{5}e^{2t}\sin t - \frac{1}{5}e^{2t}\cos t\right] + c$$

$$e^t v = 2e^{2t}\sin t - e^{2t}\cos t + c$$

Using the initial conditions, that “A small object is initially at rest”

$$1 \times 0 = 2 \times 1 \times 0 - 1 \times 1 + c \text{ gives } c = 1$$

$$e^t v = 2e^{2t}\sin t - e^{2t}\cos t + 1$$

Now to find the velocity when $t = 2\pi$...

$$e^{2\pi} v = 2e^{4\pi}\sin(2\pi) - e^{4\pi}\cos(2\pi) + 1$$

$$= 1 - e^{4\pi}$$

$$v = e^{-2\pi} - e^{2\pi} \qquad \text{(About } -535\text{ m/s)}$$

$$\text{speed} = |v|$$

$$= e^{2\pi} - e^{-2\pi}$$

[6 BONUS marks]