**Push The Pace #3**
**Answers**

**Further A-Level Pure Mathematics**
**Push The Pace Revision Papers**

**Answer 1**

$$V = \pi \int_0^{\frac{\pi}{2}} sec^2\left(\frac{x}{2}\right) dx$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \left(\frac{1}{2}\right) sec^2\left(\frac{x}{2}\right) dx \qquad \text{setting up a “chain rule backwards”}$$

$$= 2\pi \left[ tan\left(\frac{x}{2}\right) \right]_0^{\frac{\pi}{2}}$$

$$= 2\pi \left[ tan\left(\frac{\pi}{4}\right) - tan(0) \right]$$

$$= 2\pi$$

**[ 3 marks ]**

**Answer 2**

**( i )**

$$\text{LHS} = \mathbf{A}^2$$

$$= \begin{pmatrix} 5 & 4 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} 5 & 4 \\ -3 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 13 & 12 \\ -9 & -8 \end{pmatrix}$$

$$\text{RHS} = 3\mathbf{A} - 2\mathbf{I}$$

$$= 3\begin{pmatrix} 5 & 4 \\ -3 & -2 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 13 & 12 \\ -9 & -8 \end{pmatrix}$$

$$= \text{LHS} \qquad \therefore \text{ Verified}$$

**[ 4 marks ]**

**( ii )**

$$\mathbf{A}^2 = 3\mathbf{A} - 2\mathbf{I}$$

$$\mathbf{A}^{-1}\mathbf{A}^2 = 3\,\mathbf{A}^{-1}\mathbf{A} - 2\,\mathbf{A}^{-1}\mathbf{I} \qquad \text{Pre-multiply through by } \mathbf{A}^{-1}$$

$$\mathbf{A} = 3\,\mathbf{I} - 2\,\mathbf{A}^{-1}$$

$$\mathbf{A}^{-1} = -\frac{1}{2}\mathbf{A} + \frac{3}{2}\mathbf{I}$$

**[ 3 marks ]**

**Answer 3**

**( a )** Product of the roots: $\alpha \times \frac{1}{\alpha} \times \beta = 2$ so $\beta = 2$

Sum of the roots:

$$\alpha + \frac{1}{\alpha} + \beta = 1$$

$$\alpha + \frac{1}{\alpha} + 1 = 0 \Rightarrow \text{ Note that } \alpha + \frac{1}{\alpha} = -1 \; **$$

$$\alpha^2 + \alpha + 1 = 0$$

$$\alpha = \frac{-1 \pm \sqrt{1 - 4 \times 1 \times 1}}{2}$$

$$= \frac{-1 \pm \sqrt{-3}}{2}$$

$$= -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\,\mathrm{i}$$

The roots are, 2, $\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\,\mathrm{i}\right)$ and $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\,\mathrm{i}\right)$

**[ 6 marks ]**

**( b )** $k = \alpha \times \frac{1}{\alpha} + \alpha\beta + \frac{1}{\alpha} \times \beta$

$$= 1 + 2\left(\alpha + \frac{1}{\alpha}\right)$$

$$= 1 + 2 \times (-1) \text{ from } **$$

$$= -1$$

**[ 2 marks ]**

**Answer 4**

**( a )** The $z^2$ term has a coefficient of $(-2\mathrm{i})$ which is not real.
As the roots only occur in conjugate pairs if all the coefficients of the polynomial are real numbers, Stephen is not correct.

**[ 1 mark ]**

**( b )** A cubic has three root and we are told two. Let the third be $\alpha$.

The sum of the roots: $1 + \mathrm{i} - 1 + \alpha = 2\mathrm{i} \Rightarrow \alpha = \mathrm{i}$

The product of the roots : $(1 + \mathrm{i})(-1)(\mathrm{i}) = -q \Leftrightarrow q = -1 + \mathrm{i}$

Sum of product pairs:

$$(1 + \mathrm{i})(-1) + (-1)(\mathrm{i}) + (\mathrm{i})(1 + \mathrm{i}) = p$$

$$-1 - \mathrm{i} - \mathrm{i} + \mathrm{i} - 1 = p$$

$$p = -2 - \mathrm{i}$$

**[ 5 marks ]**

**Answer 5**

$$2z - 5\mathrm{i}\,z^* = 12$$

$$2(a + b\mathrm{i}) - 5\mathrm{i}(a - b\mathrm{i}) = 12$$

$$2a + 2b\mathrm{i} - 5a\mathrm{i} + 5b\,\mathrm{i}^2 = 12$$

$$2a - 5b + 2b\mathrm{i} - 5a\mathrm{i} = 12$$

$$\text{so} \begin{Bmatrix} 2a - 5b = 12 \\ -5a + 2b = 0 \end{Bmatrix}$$

Solve these simultaneously: $10a - 25b = 60$

ADD $-10a + 4b = 0$

$$-21b = 60$$

$$b = -\frac{20}{7}$$

$$2a = 12 + 5\left(-\frac{20}{7}\right)$$

$$a = -\frac{8}{7}$$

$$\therefore\ z = -\frac{8}{7} - \frac{20}{7}\mathrm{i}$$

**[ 4 marks ]**

**Answer 6**

**( a )** $r\,cosh(\theta + \alpha) = r\,cosh\,\theta\,cosh\,\alpha + r\,sinh\,\theta\,sinh\,\alpha$

$r\,cosh(\theta + \alpha) = 5\,cosh\,\theta + 3\,sinh\,\theta$

$$\text{so } \frac{r\,sinh\,\alpha}{r\,cosh\,\alpha} = tanh\,\alpha = \frac{3}{5}$$

$$\alpha = artanh\left(\frac{3}{5}\right)$$

$$\alpha = 0.693 \qquad r = \sqrt{5^2 - 3^2} = 4$$

$$\therefore\ 5\,cosh\,\theta + 3\,sinh\,\theta = 4\,cosh(\theta + 0.693)$$

**[ 4 marks ]**

**( b )**

$$5\,cosh\,\theta + 3\,sinh\,\theta = 10$$

$$4\,cosh(\theta + 0.693) = 10$$

$$cosh(\theta + 0.693) = 2.5$$

$$\theta + 0.693 = \pm 1.567$$

$$\theta = 0.874, \quad -2.26$$

**[ 4 marks ]**

**Answer 7**

$$\begin{aligned} I_n &= \int_0^{\frac{\pi}{4}} tan^n x\, dx \\ &= \int_0^{\frac{\pi}{4}} tan^{n-2} x \; tan^2 x \; dx \\ &= \int_0^{\frac{\pi}{4}} tan^{n-2} x \left( sec^2 x - 1 \right) dx \\ &= \int_0^{\frac{\pi}{4}} sec^2 x \; tan^{n-2} x \; dx - \int_0^{\frac{\pi}{4}} tan^{n-2} x \; dx \\ &= \int_0^{\frac{\pi}{4}} sec^2 x \; tan^{n-2} x \; dx - I_{n-2} \\ &= \left[ \frac{tan^{n-1} x}{n - 1} \right]_0^{\frac{\pi}{4}} - I_{n-2} \\ &= \left[ \frac{1}{n-1} \right] - I_{n-2} \qquad \square \end{aligned}$$

**[ 5 marks ]**

**( b )**

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \left( 3 + tan^2 x \right)^2 dx &= \int_0^{\frac{\pi}{4}} 9 + 6\, tan^2 x + tan^4 x \; dx \\ &= 9\, I_0 + 6\, I_2 + I_4 \\ &= 9\, I_0 + 6\, I_2 + \left( \frac{1}{4 - 1} \right) - I_2 \\ &= 9\, I_0 + 5\, I_2 + \frac{1}{3} \\ &= 9\, I_0 + 5 \times \left( \frac{1}{2 - 1} \right) - 5\, I_0 + \frac{1}{3} \\ &= 4\, I_0 + 5 + \frac{1}{3} \\ &= 4 \times \frac{\pi}{4} + \frac{16}{3} \\ &= \pi + \frac{16}{3} \end{aligned}$$

**[ 7 marks ]**