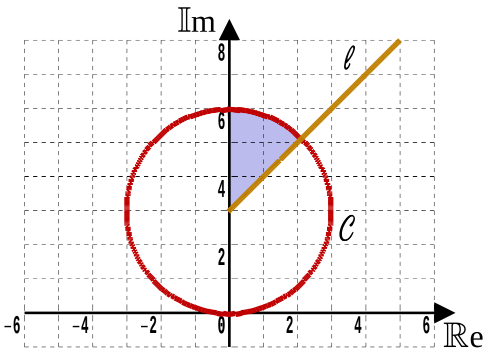
**Push The Pace #4**
**Answers**

**Further A-Level Pure Mathematics**
**Push The Pace Revision Papers**

**Answer 1**




**( i )** The half-line $\ell$ has equation $z - 3\text{i} = arg\left(\frac{\pi}{4}\right)$

The circle $\mathcal{C}$ has equation, $|z - 3\text{i}| = 3$

**[ 4 marks ]**

**( ii )** The region shaded could be specified by,

$$\left\{ arg\left(\frac{\pi}{4}\right) \leqslant z - 3\text{i} \leqslant arg\left(\frac{\pi}{2}\right) \right\} \cap \{ |z - 3\text{i}| \leqslant 3 \}$$

**[ 3 marks ]**

**Answer 2**

**( a )** Let $x = r\cos\theta$ and $y = r\sin\theta$ in which case $x^2 + y^2 = r^2$

In consequence, $\left(x^2 + y^2\right)^2 = 2c^2 xy$ becomes,

$$\left(r^2\right)^2 = 2c^2\,(r\cos\theta)(r\sin\theta)$$

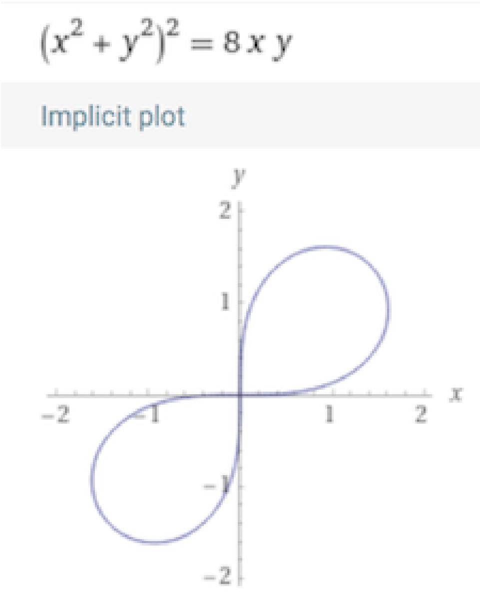
Divide through by $r^2$

$$r^2 = c^2\,(2\sin\theta\cos\theta)$$

$r^2 = c^2\sin 2\theta$ is the polar equation of the curve

**[ 2 marks ]**

**( b )** Here is the plot from Wolfram Alpha with $c = 2$




**[ 3 marks ]**

**( c )**

$$\mathrm{I} = \frac{1}{2}\int r^2\,d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} c^2\sin 2\theta\,d\theta$$

$$= \frac{c^2}{4}\int_0^{\frac{\pi}{2}} 2\sin 2\theta\,d\theta$$

$$= \frac{c^2}{4}\left[-\cos 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{c^2}{4}\times(+2)$$ and so the area is $\frac{c^2}{2}$

**[ 3 marks ]**

**Answer 3**

**( a )**

$$|\mathbf{A}| = 3$$

$$\begin{vmatrix} 2 & 1 & 4 \\ -3 & p & 2 \\ -1 & -2 & 5 \end{vmatrix} = 3$$

Expanding along the middle row...

$$3\begin{vmatrix} 1 & 4 \\ -2 & 5 \end{vmatrix} + p\begin{vmatrix} 2 & 4 \\ -1 & 5 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ -1 & -2 \end{vmatrix} = 3$$

$$3 \times 13 + 14p + 6 = 3$$

$$14p = -42$$

$$p = -3$$

**[ 3 marks ]**

**( b )** $\mathbf{AB} = \begin{pmatrix} 2 & 1 & 4 \\ -3 & -3 & 2 \\ -1 & -2 & 5 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ q & 3 \\ 4 & 0 \end{pmatrix}$

$$= \begin{pmatrix} q + 16 & 5 \\ -3q + 8 & -12 \\ -2q + 20 & -7 \end{pmatrix}$$

**[ 2 marks ]**

**( c )** **AB** is not a square matrix.
Inverses are only defined for square matrices.
$\therefore$ **AB** has no inverse.

**[ 1 mark ]**

**Answer 4**

**( a )** From the quadratic we get that...

$$p x^2 + qx + r = 0$$

$$x^2 + \frac{q}{p} x + \frac{r}{p} = 0$$

$$\text{Sum: } \alpha + \beta = -\frac{q}{p} \qquad \text{Product: } \alpha\beta = \frac{r}{p}$$

For the new cubic equation we require...

$$\text{Sum: } \alpha + \beta + (\alpha + \beta) = 2(\alpha + \beta)$$

$$= -\frac{2q}{p}$$

$$\text{Sum of Product Pairs: } \alpha\beta + \beta(\alpha + \beta) + (\alpha + \beta) = \alpha\beta + (\alpha + \beta)^2$$

$$= \frac{r}{p} + \frac{q^2}{p^2}$$

$$\text{Product: } \alpha\beta(\alpha + \beta) = -\frac{qr}{p^2}$$

A cubic with roots $\alpha$, $\beta$ and $(\alpha + \beta)$ is,

$$x^3 + \left(\frac{2q}{p}\right)x^2 + \left(\frac{r}{p} + \frac{q^2}{p^2}\right)x + \frac{qr}{p^2} = 0$$

**[ 6 marks ]**

**( b )**

$$p x^2 - qx - r = 0$$

$$x^2 - \frac{q}{p} x - \frac{r}{p} = 0$$

$$\text{Sum: } 2\alpha + \gamma = \frac{q}{p} \qquad \text{Product: } 2\alpha\gamma = -\frac{r}{p}$$

$$2\alpha\gamma = -\alpha\beta$$

$$\beta = -2\gamma$$

**[ 3 marks ]**

**Answer 5**

**( a )**

$$
\begin{aligned}
\text{LHS} &= \frac{\sinh\theta}{1+\cosh\theta} + \frac{1+\cosh\theta}{\sinh\theta} \\
&= \frac{\sinh^2\theta + (1+\cosh\theta)^2}{(1+\cosh\theta)\sinh\theta} \\
&= \frac{\sinh^2\theta + 1 + 2\cosh\theta + \cosh^2\theta}{(1+\cosh\theta)\sinh\theta} \\
&= \frac{\sinh^2\theta + \cosh^2\theta - \sinh^2\theta + 2\cosh\theta + \cosh^2\theta}{(1+\cosh\theta)\sinh\theta} \\
&= \frac{2\cosh^2\theta + 2\cosh\theta}{(1+\cosh\theta)\sinh\theta} \\
&= \frac{2\cosh\theta(\cosh\theta + 1)}{(1+\cosh\theta)\sinh\theta} \\
&= \frac{2\cosh\theta}{\sinh\theta} \\
&= 2\coth\theta \\
&= \text{RHS} \qquad \square
\end{aligned}
$$

**[ 4 marks ]**

**( b )**

$$
\begin{aligned}
\frac{\sinh\theta}{1+\cosh\theta} + \frac{1+\cosh\theta}{\sinh\theta} &= 4 \\
2\coth\theta &= 4 \qquad \text{from part (a)} \\
\frac{2}{\tanh\theta} &= 4 \\
\tanh\theta &= \frac{1}{2} \\
(\theta &= 0.5493...) \\
\theta &= \operatorname{artanh}\left(\frac{1}{2}\right) \\
&= \frac{1}{2}\ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right) \\
&= \frac{1}{2}\ln 3
\end{aligned}
$$

**[ 2 marks ]**

**Answer 6**

$$2z - \mathrm{i}\, z^* = \frac{2 + \mathrm{i}}{1 - \mathrm{i}}$$

$$2(a + b\mathrm{i}) - \mathrm{i}(a - b\mathrm{i}) = \frac{(2 + \mathrm{i})}{(1 - \mathrm{i})} \times \frac{(1 + \mathrm{i})}{(1 + \mathrm{i})}$$

$$2a + 2b\mathrm{i} - a\mathrm{i} + b\,\mathrm{i}^2 = \frac{2 + 3\mathrm{i} + \mathrm{i}^2}{1 - \mathrm{i}^2}$$

$$(2a - b) + (-a + 2b) = \frac{1}{2} + \frac{3}{2}\mathrm{i}$$

$$(4a - 2b) + (-2a + 4b) = 1 + 3\mathrm{i}$$

Equating real parts and equating imaginary parts...

$$\left.\begin{aligned} 4a - 2b &= 1 \\ -2a + 4b &= 3 \end{aligned}\right\} = \left\{\begin{aligned} 8a - 4b &= 2 \\ -2a + 4b &= 3 \end{aligned}\right.$$

$$6x \qquad = 5$$

$$a = \frac{5}{6} \text{ and } b = \frac{7}{6}$$

$$\therefore z = \frac{5}{6} + \frac{7}{6}\mathrm{i}$$

**[ 4 marks ]**

**Answer 7**

**( a )** If $y = \lambda x \sin 5x$ then $\dfrac{dy}{dx} = 5\lambda x \cos 5x + \lambda \sin 5x$

$$\frac{d^2y}{dx^2} = -25\lambda x \sin 5x + 5\lambda \cos 5x + 5\lambda \cos 5x$$

$$\frac{d^2y}{dx^2} + 25y = 3\cos 5x$$

$$-25\lambda x \sin 5x + 5\lambda \cos 5x + 5\lambda \cos 5x + 25\left(\lambda x \sin 5x\right) = 3\cos 5x$$

$$10\lambda \cos 5x = 3\cos 5x$$

$$\lambda = \frac{3}{10}$$

**[ 4 Bonus Marks ]**

**( b )** Consider $\dfrac{d^2y}{dx^2} + 25y = 0$ which has auxiliary equation $m^2 + 25 = 0$

The complex roots of $\pm 5\mathrm{i}$ lead to the complementary function being,

$$y = A\cos 5x + B\sin 5x$$

$\therefore$ The general solution is $y = A\cos 5x + B\sin 5x + \dfrac{3}{10}x\sin 5x$

**[ 3 Bonus Marks ]**

**( c )** Using the initial condition that $x = 0,\ y = 0$ gives,

$$0 = A$$

What's left of the general solution for these initial conditions is,

$$y = B\sin 5x + \frac{3}{10}x\sin 5x$$

$$\frac{dy}{dx} = 5B\cos 5x + \frac{3 \times 5}{10}x\cos 5x + \frac{3}{10}\sin 5x$$

$$= \frac{3}{10}\sin 5x + \left(5B + \frac{3}{2}x\right)\cos 5x$$

Using the initial condition that $x = 0,\ \frac{dy}{dx} = 5$ gives,

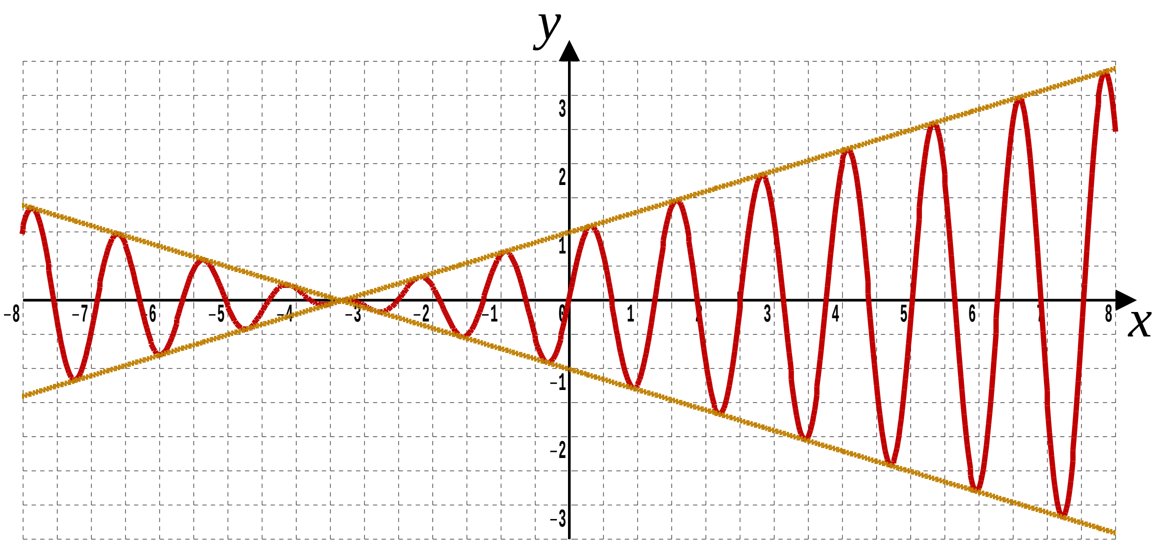
$$5 = 5B \qquad \text{so } B = 1$$

The particular solution is $y = \sin 5x + \frac{3}{10}x\sin 5x$

$$= \left(1 + \frac{3}{10}x\right)\sin 5x$$

**[ 5 bonus marks ]**

**( d )** This is a sine wave $\left(\text{with period } \frac{2\pi}{5}\right)$ with envelope $y = \frac{3}{10}x + 1$




**[ 2 bonus marks ]**