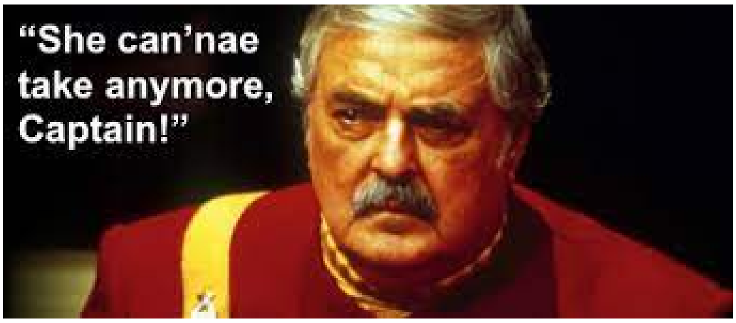
Further Pure A-Level Mathematics
Compulsory Course Components
Core 1 and Core 2

PUSH THE PACE

Further A-Level
Mathematics Revision



Push The Pace #1

You have thirty-five minutes to answer seven examination questions

Marks Available : 40 (+ 6 bonus)

Further A-Level Pure Mathematics
Push The Pace Revision Papers

Question 1

Further A-Level Examination Question from October 2021, Paper 1, Q6 (OCR)

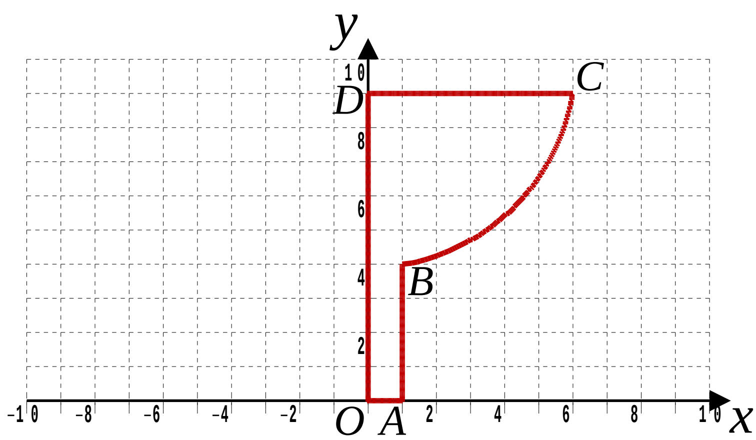
O is the origin of a coordinate system whose units are cm.

The points A, B, C and D have coordinates (1, 0), (1, 4), (6, 9) and (0, 9) respectively.

The arc BC is part of the curve with equation $x^2 + (y - 10)^2 = 37$

The closed shape $OABCD$ is formed, in turn, from the line segments OA and AB, the arc BC and the line segments CD and DO (see diagram).

A funnel can be modelled by rotating $OABCD$ by 2π radians about the y-axis.



Find the volume of the funnel according to the model.

[3 marks]

Question 2

Further A-Level Examination Question from June 2020, Paper 2, Q4 (AQA)

The matrices **A** and **B** are defined as follows,

$$\mathbf{A} = \begin{pmatrix} x+1 & 2 \\ x+2 & -3 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} x-4 & x-2 \\ 0 & -2 \end{pmatrix}$$

Show that there is a value of x for which $\mathbf{AB} = k\mathbf{I}$, where $\mathbf{I}$ is the 2×2 identity matrix and k is an integer to be found.

[3 marks]

Question 3

Further A-Level Examination Question from June 2019, Paper 2, Q4 (AQA)

The positive integer k is such that, $\displaystyle\sum_{r=1}^{k} (3r - k) = 90$

Find the value of k

[3 marks]

Question 4

Further A-Level Examination Question from June 2022, Paper 4, Q7 (WJEC)

(a) Express $4x^2 + 10x - 24$ in the form $a(x + b)^2 + c$, where a, b and c are constants whose values are to be found.

[3 marks]

(b) Hence evaluate the integral $\int_3^5 \frac{6}{\sqrt{4x^2 + 10x - 24}}\,dx$

Give your answer correct to 3 decimal places.

[5 marks]

Question 5

Further A-Level Examination Question from June 2022, Paper 4, Q12 (WJEC)

Find the solution of the differential equation,

$$3\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 2y = 8 + 6x - 2x^2$$

where $y = 6$ and $\frac{dy}{dx} = 5$ when $x = 0$

[12 marks]

Question 6

Further A-Level Examination Question from May 2020, Paper 1, Q8 (AQA)

The three roots of the equation,

$$4x^3 - 12x^2 - 13x + k = 0$$

where k is a constant, form an arithmetic sequence.

Find the roots of the equation.

[6 marks]

Question 7

Further A-Level Examination Question from June 2022, Paper 4, Q5 (WJEC)

(a) Determine the number of solutions of the equations,

$$\begin{aligned} x + 2y &= 3 \\ 2x - 5y + 3z &= 8 \\ 6y - 2z &= 0 \end{aligned}$$

[5 marks]

(b) Each of the three equations in part (a) has the geometric interpretation of being a three dimensional plane. Backed up by a rigorous analysis determine the configuration of the three planes.

[6 BONUS marks]

Push The Pace #1
Answers

Answer 1

For the cylinder formed by rotating the stem of the funnel about the y-axis,

$$V_{CYLINDER} = \pi \times 1^2 \times 4$$
$$= 4\pi$$

For the bowl formed by rotating the arc of the circle about the y-axis,

$$V_{BOWL} = \pi \int_4^9 37 - (y - 10)^2 \, dy$$
$$= \pi \int_4^9 37 - y^2 + 20y - 100 \, dy$$
$$= \pi \int_4^9 -y^2 + 20y - 63 \, dy$$
$$= \pi \left[-\frac{y^3}{3} + 10y^2 - 63y \right]_4^9$$
$$= \pi \left[-243 + 810 - 567 + \frac{64}{3} - 160 + 252 \right]$$
$$= \frac{340\pi}{3}$$

Total Volume is $\frac{352\pi}{3}$ cm^3 (About 368.6 cm^3)

[3 marks]

Answer 2

$$\mathbf{AB} = k\mathbf{I}$$

$$\begin{pmatrix} x+1 & 2 \\ x+2 & -3 \end{pmatrix}\begin{pmatrix} x-4 & x-2 \\ 0 & -2 \end{pmatrix} = k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} (x+1)(x-4) & x^2 - x - 6 \\ (x+2)(x-4) & x^2 + 2 \end{pmatrix} = k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} (x+1)(x-4) & (x+2)(x-3) \\ (x+2)(x-4) & x^2 + 2 \end{pmatrix} = k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus, $x = -2$ (to get the zeros) and the multiplication is then,

$$\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} = k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So $k = 6$ when $x = -2$

[3 marks]

Answer 3

$$\sum_{r=1}^{k} (3r - k) = 90$$

$$3 \sum_{r=1}^{k} r - k \sum_{r=1}^{k} 1 = 90$$

$$3 \times \frac{k(k+1)}{2} - k^2 = 90$$

$$3k(k+1) - 2k^2 = 180$$

$$k^2 + 3k - 180 = 0$$

$$(k+15)(k-12) = 0$$

Either $k = -15$ (reject) or $k = 12$

The only answer is $k = 12$

[3 marks]

Answer 4

(a) $4x^2 + 10x - 24 = 4\left[x^2 + \frac{5}{2}x\right] - 24$

$$= 4\left[\left(x + \frac{5}{4}\right)^2 - \frac{25}{16}\right] - 24$$

$$= 4\left(x + \frac{5}{4}\right)^2 - \frac{25}{4} - 24$$

$$= 4\left(x + \frac{5}{4}\right)^2 - \frac{121}{4} \qquad \left(a = 4,\ b = \frac{5}{4},\ c = -\frac{121}{4}\right)$$

[3 marks]

(b)

$$= 4\left[\left(x + \frac{5}{4}\right)^2 - \frac{121}{16}\right]$$

$$= 4\left[\left(x + \frac{5}{4}\right)^2 - \left(\frac{11}{4}\right)^2\right]$$

Now to use the formulae booklet (page 12) where we are told that,

$$\int \frac{1}{\sqrt{x^2 - a^2}}\, dx = arcosh\left(\frac{x}{a}\right),\ ln\left\{x + \sqrt{x^2 - a^2}\right\} + c \quad \text{for } x > a$$

By all means use the first result, we're going for the second...

$$\int_3^5 \frac{6}{\sqrt{4x^2 + 10x - 24}}\, dx$$

$$= \frac{1}{2}\int_3^5 \frac{6}{\sqrt{\left(x + \frac{5}{4}\right)^2 - \left(\frac{11}{4}\right)^2}}\, dx$$

$$= 3\left[ln\left\{x + \frac{5}{4} + \sqrt{\left(x + \frac{5}{4}\right)^2 - \left(\frac{11}{4}\right)^2}\right\}\right]_3^5$$

$$= 3\left[ln\left\{\frac{25}{4} + \sqrt{\left(\frac{25}{4}\right)^2 - \left(\frac{11}{4}\right)^2}\right\} - ln\left\{\frac{17}{4} + \sqrt{\left(\frac{17}{4}\right)^2 - \left(\frac{11}{4}\right)^2}\right\}\right]$$

$$= 3\left[ln\left\{\frac{25}{4} + \frac{6\sqrt{14}}{4}\right\} - ln\left\{\frac{17}{4} + \frac{2\sqrt{42}}{4}\right\}\right]$$

$$= 3\,ln\left\{\frac{25 + 6\sqrt{14}}{17 + 2\sqrt{42}}\right\}$$

$$= 1.379$$

[5 marks]

Answer 5

Begin by solving the homogeneous equation, $3\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 2y = 0$

This has auxiliary equation, $3m^2 + 5m - 2 = 0$

$$(3m - 1)(m + 2) = 0$$

Either $m = \frac{1}{3}$ or $m = -2$ (Two distinct real roots)

The general solution is $y = Ae^{\frac{1}{3}x} + Be^{-2x}$

For the particular integral try $y = Ux^2 + Vx + W$

in which case $\frac{dy}{dx} = 2Ux + V$ and $\frac{d^2y}{dx^2} = 2U$

Substitute these into the question's differential equation...

$$3 \times 2U + 5(2Ux + V) - 2\left(Ux^2 + Vx + W\right) = 8 + 6x - 2x^2$$

$$6U + 10Ux + 5V - 2Ux^2 - 2Vx - 2W = 8 + 6x - 2x^2$$

$$(6U + 5V - 2W) + (10U - 2V)x - 2Ux^2 = 8 + 6x - 2x^2$$

Match up coefficients of x^2 : $U = 1$

Match up coefficients of x : $V = 2$

Terms independent of x : $6 + 10 - 2W = 8$ gives $W = 4$

$\therefore$ The particular integral is $y = x^2 + 2x + 4$

The general solution is $y = Ae^{\frac{1}{3}x} + Be^{-2x} + x^2 + 2x + 4$

from which $\frac{dy}{dx} = \frac{A}{3}e^{\frac{1}{3}} - 2Be^{-2} + 2x + 2$

From the boundary conditions, $y = 6$ and $\frac{dy}{dx} = 5$ when $x = 0$

$$6 = A + B + 4 \text{ and } 5 = \frac{A}{3} - 2B + 2$$

Solving these simultaneously...

$$\begin{aligned} A + B &= 2 \\ \text{SUBTRACT} \quad A - 6B &= 9 \\ \hline 7B = -7 \text{ so } B = -1 &\text{ and } A = 3 \end{aligned}$$

The particular solution is $y = 3e^{\frac{1}{3}x} - e^{-2x} + x^2 + 2x + 4$

[12 marks]

Answer 6

You could use the various rules for sum or products of roots.
Here is a "first principles" approach.

$$4x^3 - 12x^2 - 13x + k = 0$$

Divide through by 4

$$x^3 - 3x^2 - \frac{13}{4}x + \frac{k}{4} = 0$$

Let the roots be $(a - d)$, a, $(a + d)$ which are in arithmetic progression.

(For some constant a)

$$\text{In consequence, } (x - a + d)(x - a)(x - a - d) = 0$$

$$\left((x - a)^2 - d^2 \right)\left(x - a \right) = 0$$

$$\left(x^2 - 2ax + a^2 - d^2 \right)\left(x - a \right) = 0$$

$$x^3 - 2ax^2 + \left(a^2 - d^2 \right)x$$

$$- ax^2 + 2a^2x - a^3 + ad^2 = 0$$

$$x^3 - 3ax^2 + \left(3a^2 - d^2 \right)x - a^3 + ad^2 = 0$$

Match up coefficients of x^2 : $a = 1$

Match up coefficients of x : $3a^2 - d^2 = -\frac{13}{4}$ so $d^2 = 3 + \frac{13}{4}$

$$d = \pm \frac{5}{2}$$

So the roots are, $x = -\frac{3}{2}$, $x = 1$, and $x = \frac{7}{2}$

[6 marks]

Answer 7

(a) In matrix form the equations are, $\begin{pmatrix} 1 & 2 & 0 \\ 2 & -5 & 3 \\ 0 & 6 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \\ 0 \end{pmatrix}$

However, the 3×3 matrix has a determinant of zero.
To see this either use your calculator or, expand along the upper row,

$$\begin{vmatrix} 1 & 2 & 0 \\ 2 & -5 & 3 \\ 0 & 6 & -2 \end{vmatrix} = (1)\begin{vmatrix} -5 & 3 \\ 6 & -2 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ 0 & -2 \end{vmatrix} + 0\begin{vmatrix} 2 & -5 \\ 0 & 6 \end{vmatrix}$$

$$= 10 - 18 + 8$$

$$= 0 \quad \text{as claimed.}$$

This means that there is no unique solutions and so the planes do not meet at a single point.

[5 marks]

(b) Look at the normal vectors which are $\begin{pmatrix}1\\2\\0\end{pmatrix}$, $\begin{pmatrix}2\\-5\\6\end{pmatrix}$ and $\begin{pmatrix}0\\3\\2\end{pmatrix}$ but none of these are parallel.

Multiply the upper row by 15, the middle by 6 and the lower by 5

$$\begin{pmatrix} 15 & 30 & 0 \\ 12 & -30 & 18 \\ 0 & 30 & -10 \end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}45\\48\\0\end{pmatrix}$$

Combine upper and middle equations,

$$\begin{array}{rrcl} & 15x + 30y + 0z & = & 45 \\ \text{ADD} & 12x - 30y + 18z & = & 48 \\ \hline & 27x + 18z & = & 93 \\ & 9x + 6z & = & 31 \quad * \end{array}$$

Combine lower and middle equations,

$$\begin{array}{rrcl} & 0x + 30y - 10z & = & 0 \\ \text{ADD} & 12x - 30y + 18z & = & 48 \\ \hline & 12x + 8z & = & 48 \\ & 3x + 2z & = & 12 \quad * \end{array}$$

As the two starred equations are not consistent, the three planes form a prism.

[6 BONUS marks]

Push The Pace #2

You have thirty-five minutes to answer seven examination questions

Marks Available : 40 (+ 6 bonus)

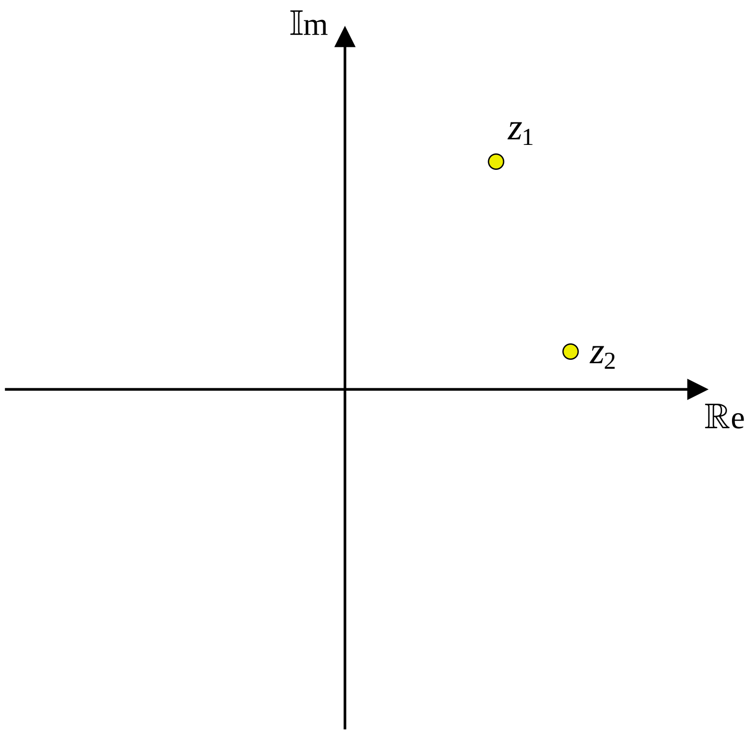
Further A-Level Pure Mathematics

Push The Pace Revision Papers

Question 1

Further AS-Level Examination Question from October 2020, Paper 1, Q2 (OCR)

The Argand diagram shows two complex numbers z_1 and z_2



(a) Mark points representing each of the following complex numbers,

- z_1^*
- $z_2 - z_1$

[2 marks]

(b) In the case where $z_1 = 1 + 2\text{i}$ and $z_2 = 3 + \text{i}$, find $\dfrac{z_2 - z_1}{z_1^*}$ in the form $a + b\text{i}$, where a and b are real numbers.

[2 marks]

Question 2

Further A-Level Examination Question from June 2022, Paper 4, Q1 (WJEC)

A function f has domain $(-\infty,\ \infty)$ and is defined by $f(x) = cosh^3 x - 3\, cosh\, x$

(a) Show that the graph of $y = f(x)$ has only one stationary point.

[4 marks]

(b) Find the nature of this stationary point.

[3 marks]

(c) State the largest possible range of $f(x)$

[1 mark]

Question 3

Further A-Level Examination Question from June 2019, Paper 2, Q2 (WJEC)

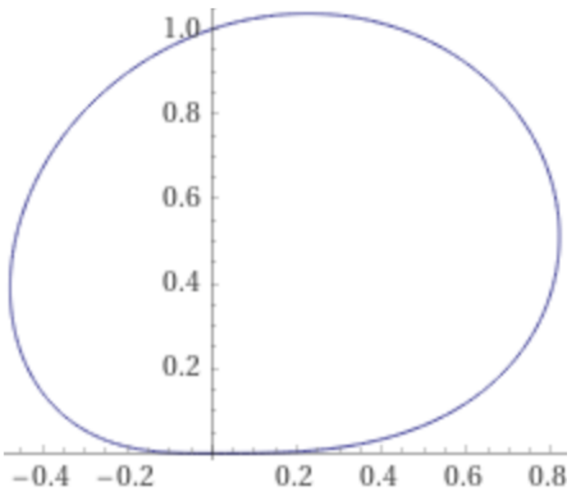
When plotted on an Argand diagram, the four fourth roots of the complex number $9 - 3\sqrt{3}\,\mathrm{i}$ lie on a circle. Find the equation of this circle.

[4 marks]

Question 4

Further A-Level Examination Question from June 2019, Paper 2, Q9 (OCR)

The diagram shows the curve $r = \sqrt{\sin\theta}\, e^{\frac{1}{3}\cos\theta}$ for $0 \leqslant \theta \leqslant \pi$



(a) Find the exact area enclosed by the curve.

[4 marks]

(b) Show that the greatest value of r on the curve is $\sqrt{\dfrac{\sqrt{3}}{2}}\ e^{\frac{1}{6}}$

[7 marks]

Question 5

Further A-Level Examination Question from June 2019, Paper 2, Q10 (OCR)

(a) Use differentiation to find the first two non-zero terms of the Maclaurin expansion of $ln\left(\frac{1}{2} + cos\,x\right)$

[4 marks]

(b) By considering the root of the equation $ln\left(\frac{1}{2} + cos\,x\right) = 0$ deduce that $\pi \approx 3\sqrt{3\,ln\left(\frac{3}{2}\right)}$

[3 marks]

Question 6

Further A-Level Examination Question from June 2020, Paper 2, Q12 (AQA)

(a) Given that $I = \int_a^b e^{2t} \sin t \, dt$, show that $I = \left[q e^{2t} \sin t + r e^{2t} \cos t \right]_a^b$

where q and r are rational numbers to be found.

[6 marks]

(b) A small object is initially at rest. The subsequent motion of the object is modelled by the differential equation,

$$\frac{dv}{dt} + v = 5\,e^{t}\,sin\,t$$

where v is the velocity at time t

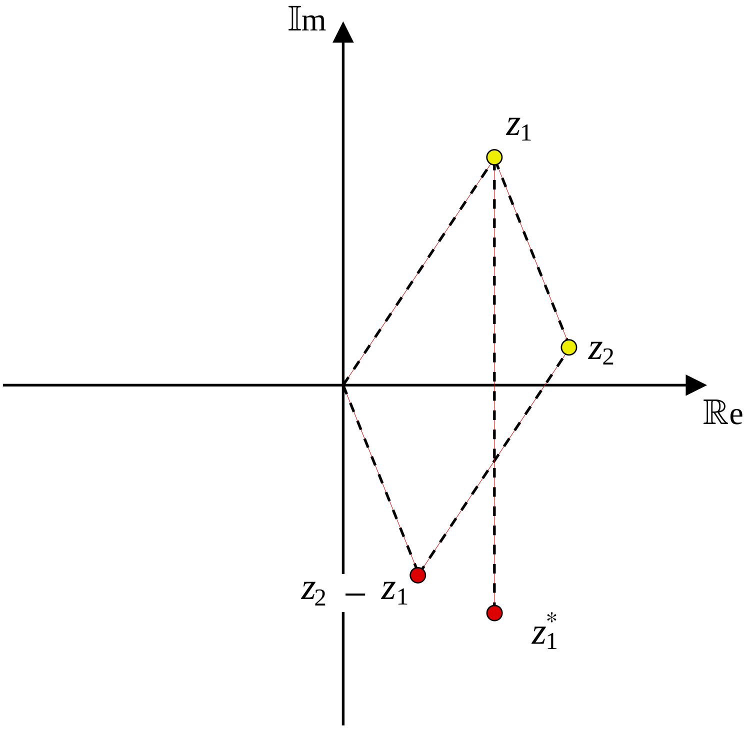
Find the speed of the object when $t = 2\pi$, giving your answer in exact form.

[6 BONUS marks]

Push The Pace #2
Answers

Answer 1

(a)



[2 marks]

(b)

$$\frac{z_2 - z_1}{z_1^*} = \frac{3 + i - 1 - 2i}{1 - 2i}$$

$$= \frac{2 - i}{1 - 2i} \times \frac{1 + 2i}{1 + 2i}$$

$$= \frac{2 - i + 4i - 2i^2}{1 - 4i^2}$$

$$= \frac{4 + 3i}{5}$$

$$= \frac{4}{5} + \frac{3}{5}i \qquad \left(a = \frac{4}{5}, b = \frac{3}{5}\right)$$

[2 marks]

Answer 2

(a) $f\,(x) \;=\; cosh^3 x \;-\; 3\,cosh\,x$

$$f'\,(x) \;=\; 3\,cosh^2 x\,sinh\,x \;-\; 3\,sinh\,x$$

$$= \; 3\,sinh\,x\,\left(cosh^2 x \;-\; 1\right)$$

$$= \; 3\,sinh\,x\,\left(sinh^2 x\right) \qquad \text{using the identity } cosh^2 x \;-\; sinh^2 x \;=\; 1$$

$$= \; 3\,sinh^3 x$$

For a stationary point, $f'\,(x) \;=\; 0$

$$3\,sinh^3 x \;=\; 0$$

$$sinh\,x \;=\; 0$$

$$x \;=\; 0$$

So the graph of $f\,(x)$ only has one stationary point which is at the origin.

[4 marks]

(b) $f''\,(x) \;=\; 9\,sinh^2 x\,cosh\,x$

but $f''\,(0) \;=\; 0$ so this method has failed !

Instead look at the gradient either side of $x \;=\; 0$

There is no need to worry about how close to go to $x \;=\; 0$ as we know that there are no other stationary points.

$f'\,(-\,1) \;=\; 3\,sinh^3\,(-\,1) \;=\; -\,4.86,$ negative gradient

$f'\,(+\,1) \;=\; 3\,sinh^3\,(+\,1) \;=\; +\,4.87,$ positive gradient

The stationary point is a minimum.

[3 marks]

(c) We know the graph has a minimum at when $x \;=\; 0$

This has a value of $f\,(0) \;=\; cosh^3\,(0) \;-\; 3\,cosh\,(0)$

$$= \; -\,2$$

So the greatest range is $-\,2 \;\leqslant\; f\,(x) \;<\; \infty$

[1 mark]

Answer 3

We are told that $z^4 = 9 - 3\sqrt{3}\,\mathrm{i}$

Complex variable equations of the form $z^n = c$ have roots that lie on a circle with centre at the origin so we just need the radius of this circle (because we don't care where on the circle the roots are; we don't need to worry about angles).

$$\begin{aligned} |z^4| &= |9 - 3\sqrt{3}\,\mathrm{i}| \\ &= \sqrt{108} \\ \therefore z^4 &= \sqrt{108}\ arg\ \text{(angle we don't care about)} \\ z &= 108^{\frac{1}{8}}\ arg\ \text{(angles we don't care about)} \end{aligned}$$

$$\text{Equation of circle is: } x^2 + y^2 = 108^{\frac{1}{4}}$$

$$x^2 + y^2 = 3.22...$$

[4 marks]

Answer 4

(a) Area $= \frac{1}{2}\int_0^{\pi} r^2\, d\theta$

$= \frac{1}{2}\int_0^{\pi} \sin\theta\, e^{\frac{2}{3}\cos\theta}\, d\theta$

$= \left(-\frac{3}{4}\right)\int_0^{\pi} \left(-\frac{2}{3}\sin\theta\right) e^{\frac{2}{3}\cos\theta}\, d\theta$ setting up chain rule backwards

$= \left(-\frac{3}{4}\right)\left[e^{\frac{2}{3}\cos\theta}\right]_0^{\pi}$

$= \left(-\frac{3}{4}\right)\left[e^{-\frac{2}{3}} - e^{\frac{2}{3}}\right]$

$= \frac{3}{4}\left(e^{\frac{2}{3}} - e^{-\frac{2}{3}}\right)$

[4 marks]

(b) Greatest r occurs when $\frac{dr}{d\theta} = 0$

$$\frac{d}{d\theta}\left(\sqrt{\sin\theta}\; e^{\frac{1}{3}\cos\theta}\right) = 0$$

$$\sqrt{\sin\theta}\left(-\frac{1}{3}\sin\theta\right)e^{\frac{1}{3}\cos\theta} + \frac{1}{2}\,(\sin\theta)^{-\frac{1}{2}}\cos\theta\, e^{\frac{1}{3}\cos\theta} = 0$$

$$\frac{1}{6}\left(e^{\frac{1}{3}\cos\theta}\right)\left(-2\,(\sin\theta)^{\frac{3}{2}} + 3(\sin\theta)^{-\frac{1}{2}}\cos\theta\right) = 0$$

$$\left(-2\,(\sin\theta)^{\frac{3}{2}} + 3(\sin\theta)^{-\frac{1}{2}}\cos\theta\right) = 0$$

$$-2\sin^2\theta + 3\cos\theta = 0$$

$$-2\left(1 - \cos^2\theta\right) + 3\cos\theta = 0$$

$$2\cos^2\theta + 3\cos\theta - 2 = 0$$

$$(2\cos\theta - 1)(\cos\theta + 2) = 0$$

Either $\cos\theta = \frac{1}{2}$ or $\cos\theta = -2$ (no solutions)

Because of the $\sqrt{\sin\theta}$, $\sin\theta \geqslant 0$

so, from $\cos\theta = \frac{1}{2}$ we get $\sin\theta = \frac{\sqrt{3}}{2}$

Greatest value of $r = \sqrt{\sin\theta}\; e^{\frac{1}{3}\cos\theta}$

$$r_{MAX} = \sqrt{\frac{\sqrt{3}}{2}}\, e^{\frac{1}{6}}$$

[7 marks]

Answer 5

The Maclaurin Series

A given function, $f(x)$, may be written as the polynomial,

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(r)}(0)}{r!}x^r + \ldots$$

provided that $f(0)$, $f'(0)$, $f''(0)$, ..., $f^{(r)}(0)$, ... all have finite values.

(a) Let $f(x) = ln\left(\frac{1}{2} + cos\,x\right) \quad : \quad f(0) = ln\left(\frac{3}{2}\right)$

$$f'(x) = \frac{-\ sin\,x}{\frac{1}{2} + cos\,x} = \frac{-2\,sin\,x}{1 + 2\,cos\,x} \quad : \quad f'(0) = 0$$

$$f''(x) = \frac{(1 + 2\,cos\,x)(-2\,cos\,x) - (-2\,sin\,x)(-2\,sin\,x)}{(1 + 2\,cos\,x)^2}$$

$$= \frac{-2\,cos\,x - 4\,cos^2 x - 4\,sin^2 x}{(1 + 2\,cos\,x)^2}$$

$$= \frac{-2\,cos\,x - 4}{(1 + 2\,cos\,x)^2} \quad : \quad f''(0) = \frac{-6}{9} = -\frac{2}{3}$$

$$\therefore\ ln\left(\frac{1}{2} + cos\,x\right) = ln\left(\frac{3}{2}\right) - \frac{1}{3}x^2 + \ldots$$

[4 marks]

(b)

$$ln\left(\frac{1}{2} + cos\,x\right) = 0$$

$$\frac{1}{2} + cos\,x = 1$$

$$cos\,x = \frac{1}{2}$$

$$x = \frac{\pi}{3}$$

So we have that, $ln\left(\frac{3}{2}\right) - \frac{1}{3}\left(\frac{\pi}{3}\right)^2 + \ldots = 0$

$$\frac{\pi^2}{27} \approx ln\left(\frac{3}{2}\right)$$

$$\pi \approx \sqrt{27\,ln\left(\frac{3}{2}\right)}$$

$$\pi \approx 3\sqrt{3\,ln\left(\frac{3}{2}\right)}$$

[3 marks]

Answer 6

(a) In the working the limits are omitted to reduce clutter...

$$I = \int e^{2t} \, sin\, t \, dt$$

$$= e^{2t} \, (-\, cos\, t\,) - \int 2\, e^{2t} (-\, cos\, t\,)\, dt$$

$$= -\, e^{2t} \, cos\, t + 2 \int e^{2t} \, cos\, t\, dt$$

$$= -\, e^{2t} \, cos\, t + 2\, e^{2t} \, sin\, t - 2 \int 2\, e^{2t} \, sin\, t\, dt$$

$$I = -\, e^{2t} \, cos\, t + 2\, e^{2t} \, sin\, t - 4\, I$$

$$5\, I = -\, e^{2t} \, cos\, t + 2\, e^{2t} \, sin\, t$$

$$I = \left[\frac{2}{5}\, e^{2t} \, sin\, t - \frac{1}{5}\, e^{2t} \, cos\, t \right]_a^b$$

[6 marks]

(b) The question is requiring that we solve the differential equation.

$$\frac{dv}{dt} + v = 5\,e^{t}\,sin\,t$$

The equation can be solved using the integrating factor method

$\left(\text{with } P(t) = 1 \text{ and } Q(t) = 5\,e^{t}\,sin\,t\right)$

$$\int P(t)\;dt = \int 1\,dt = t$$

If there are no occurrences of the modulus function the constant of integration may be omitted, as done here.

$$I_{FACTOR} = e^{\int P(x)\,dx} = e^{t}$$

$$e^{t}\,\frac{dv}{dt} + e^{t}\,v = 5\,e^{2t}\,sin\,t$$

integrate both sides with respect to t

$$\int e^{t}\,\frac{dv}{dt} + e^{t}\,v\,dt = 5\int e^{2t}\,sin\,t\;dt$$

$$e^{t}\,v = 5\left[\frac{2}{5}\,e^{2t}\,sin\,t - \frac{1}{5}\,e^{2t}\,cos\,t\right] + c$$

$$e^{t}\,v = 2\,e^{2t}\,sin\,t - e^{2t}\,cos\,t + c$$

Using the initial conditions, that "A small object is initially at rest"

$$1 \times 0 = 2 \times 1 \times 0 - 1 \times 1 + c \text{ gives } c = 1$$

$$e^{t}\,v = 2\,e^{2t}\,sin\,t - e^{2t}\,cos\,t + 1$$

Now to find the velocity when $t = 2\pi$...

$$e^{2\pi}\,v = 2\,e^{4\pi}\,sin\,(2\pi) - e^{4\pi}\,cos\,(2\pi) + 1$$

$$= 1 - e^{4\pi}$$

$$v = e^{-2\pi} - e^{2\pi} \qquad \text{(About } - 535\text{ m/s)}$$

$$\text{speed} = |\,v\,|$$

$$= e^{2\pi} - e^{-2\pi}$$

[6 BONUS marks]

Push The Pace #3

You have thirty-five minutes to answer seven examination questions

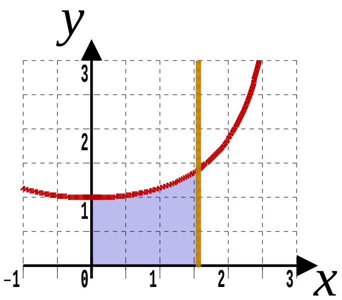
Marks Available : 40 (+ 10 bonus)

Further A-Level Pure Mathematics
Push The Pace Revision Papers

Question 1

Further A-Level Examination Question from June 2019, Paper 1, Q4 (OCR)

The graph shows the region bounded by the curve $y = sec\left(\frac{x}{2}\right)$, the x-axis, the y-axis and the line $x = \frac{\pi}{2}$



This region is rotated through 2π radians about the x-axis.
Find, in exact form, the volume of the solid of revolution generated.

[3 marks]

Question 2

Further AS-Level Examination Question from June 2016, Paper FP1. Q1 (CEA)

Let $\mathbf{A} = \begin{pmatrix} 5 & 4 \\ -3 & -2 \end{pmatrix}$ and $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(i) Verify that $\mathbf{A}^2 = 3\mathbf{A} - 2\mathbf{I}$

[4 marks]

(ii) Hence, or otherwise, express the matrix $\mathbf{A}^{-1}$ in the form $\alpha\mathbf{A} + \beta\mathbf{I}$, where α and β are real numbers.

[3 marks]

Question 3

Further A-Level Examination Question from June 2019, Paper 1, Q8 (OCR)

The roots of the equation $x^3 - x^2 + kx - 2 = 0$ are α, $\frac{1}{\alpha}$ and β

(a) Evaluate, in exact form, the roots of the equation.

[6 marks]

(b) Find k

[2 marks]

Question 4

Further AS-Level Examination Question from June 2021, Paper 1, Q8 (AQA)

Stephen is correctly told that $(1 + \mathrm{i})$ and -1 are two roots of the polynomial equation $z^3 - 2\mathrm{i}z^2 + pz + q = 0$ where p and q are complex numbers.

(a) Stephen states that $(1 - \mathrm{i})$ **must** also be a root of the equation because roots of polynomial equations occur in conjugate pairs.
Explain why Stephen's reasoning is wrong.

[1 mark]

(b) Find p and q

[5 marks]

Question 5

Further A-Level Examination Question from June 2019, Paper 1, Q4 (AQA)

Solve the equation $2z - 5\mathrm{i}z^* = 12$

[4 marks]

Question 6

Further A-Level Examination Question from June 2015, Paper FP3, Q1 (WJEC)

(a) Express $5\,cosh\,\theta + 3\,sinh\,\theta$ in the form $r\,cosh\,(\theta + \alpha)$, $r > 0$, where the values of r and α are to be found.

[4 marks]

(b) Hence solve the equation $5\,cosh\,\theta + 3\,sinh\,\theta = 10$

[4 marks]

Question 7

Further A-Level Examination Question from June 2017, Paper FP3, Q6 (WJEC)

The integral I_n is given, for $n \geqslant 0$, by $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$

(a) Show that, for $n \geqslant 2$, $I_n = \dfrac{1}{n-1} - I_{n-2}$

Hint : set up a chain rule backwards keeping in mind that the derivative of $\tan x$ is $\sec^2 x$

[2 + 3 BONUS marks]

(b) Hence determine the value of the integral, $\int_0^{\frac{\pi}{4}} \left(3 + tan^2 x \right)^2 \, dx$

leaving your answer in terms of π

[7 BONUS marks]

Answer 1

$$V = \pi \int_0^{\frac{\pi}{2}} sec^2\left(\frac{x}{2}\right) dx$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \left(\frac{1}{2}\right) sec^2\left(\frac{x}{2}\right) dx$$ setting up a "chain rule backwards"

$$= 2\pi \left[tan\left(\frac{x}{2}\right) \right]_0^{\frac{\pi}{2}}$$

$$= 2\pi\left[tan\left(\frac{\pi}{4}\right) - tan(0) \right]$$

$$= 2\pi$$

[3 marks]

Answer 2

(i)

$$\text{LHS} = \mathbf{A}^2$$

$$= \begin{pmatrix} 5 & 4 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} 5 & 4 \\ -3 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 13 & 12 \\ -9 & -8 \end{pmatrix}$$

$$\text{RHS} = 3\mathbf{A} - 2\mathbf{I}$$

$$= 3\begin{pmatrix} 5 & 4 \\ -3 & -2 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 13 & 12 \\ -9 & -8 \end{pmatrix}$$

$$= \text{LHS}$$ $\therefore$ Verified

[4 marks]

(ii)

$$\mathbf{A}^2 = 3\mathbf{A} - 2\mathbf{I}$$

$$\mathbf{A}^{-1}\mathbf{A}^2 = 3\mathbf{A}^{-1}\mathbf{A} - 2\mathbf{A}^{-1}\mathbf{I}$$ Pre-multiply through by $\mathbf{A}^{-1}$

$$\mathbf{A} = 3\mathbf{I} - 2\mathbf{A}^{-1}$$

$$\mathbf{A}^{-1} = -\frac{1}{2}\mathbf{A} + \frac{3}{2}\mathbf{I}$$

[3 marks]

Answer 3

(a) Product of the roots: $\alpha \times \frac{1}{\alpha} \times \beta = 2$ so $\beta = 2$

Sum of the roots:

$$\alpha + \frac{1}{\alpha} + \beta = 1$$

$$\alpha + \frac{1}{\alpha} + 1 = 0 \Rightarrow \text{ Note that } \alpha + \frac{1}{\alpha} = -1 \; **$$

$$\alpha^2 + \alpha + 1 = 0$$

$$\alpha = \frac{-1 \pm \sqrt{1 - 4 \times 1 \times 1}}{2}$$

$$= \frac{-1 \pm \sqrt{-3}}{2}$$

$$= -\frac{1}{2} \pm \frac{\sqrt{3}}{2} \mathrm{i}$$

The roots are, 2, $\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}\right)$ and $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}\right)$

[6 marks]

(b) $k = \alpha \times \frac{1}{\alpha} + \alpha\beta + \frac{1}{\alpha} \times \beta$

$$= 1 + 2\left(\alpha + \frac{1}{\alpha}\right)$$

$$= 1 + 2 \times (-1) \text{ from } **$$

$$= -1$$

[2 marks]

Answer 4

(a) The z^2 term has a coefficient of $(-2\mathrm{i})$ which is not real.
As the roots only occur in conjugate pairs if all the coefficients of the polynomial are real numbers, Stephen is not correct.

[1 mark]

(b) A cubic has three root and we are told two. Let the third be α.

The sum of the roots: $1 + \mathrm{i} - 1 + \alpha = 2\mathrm{i} \Rightarrow \alpha = \mathrm{i}$

The product of the roots : $(1 + \mathrm{i})(-1)(\mathrm{i}) = -q \Leftrightarrow q = -1 + \mathrm{i}$

Sum of product pairs:

$$(1 + \mathrm{i})(-1) + (-1)(\mathrm{i}) + (\mathrm{i})(1 + \mathrm{i}) = p$$

$$-1 - \mathrm{i} - \mathrm{i} + \mathrm{i} - 1 = p$$

$$p = -2 - \mathrm{i}$$

[5 marks]

Answer 5

$$2z - 5\mathrm{i}\,z^* = 12$$

$$2(a + b\mathrm{i}) - 5\mathrm{i}(a - b\mathrm{i}) = 12$$

$$2a + 2b\mathrm{i} - 5a\mathrm{i} + 5b\,\mathrm{i}^2 = 12$$

$$2a - 5b + 2b\mathrm{i} - 5a\mathrm{i} = 12$$

$$\text{so} \begin{Bmatrix} 2a - 5b = 12 \\ -5a + 2b = 0 \end{Bmatrix}$$

Solve these simultaneously: $\quad 10a - 25b = 60$

$$\text{ADD} \quad -10a + 4b = 0$$

$$-21b = 60$$

$$b = -\frac{20}{7}$$

$$2a = 12 + 5\left(-\frac{20}{7}\right)$$

$$a = -\frac{8}{7}$$

$$\therefore\ z = -\frac{8}{7} - \frac{20}{7}\mathrm{i}$$

[4 marks]

Answer 6

(a) $\quad r\cosh(\theta + \alpha) = r\cosh\theta\cosh\alpha + r\sinh\theta\sinh\alpha$

$$r\cosh(\theta + \alpha) = 5\cosh\theta + 3\sinh\theta$$

$$\text{so}\ \frac{r\sinh\alpha}{r\cosh\alpha} = \tanh\alpha = \frac{3}{5}$$

$$\alpha = \operatorname{artanh}\left(\frac{3}{5}\right)$$

$$\alpha = 0.693 \qquad r = \sqrt{5^2 - 3^2} = 4$$

$$\therefore\ 5\cosh\theta + 3\sinh\theta = 4\cosh(\theta + 0.693)$$

[4 marks]

(b)

$$5\cosh\theta + 3\sinh\theta = 10$$

$$4\cosh(\theta + 0.693) = 10$$

$$\cosh(\theta + 0.693) = 2.5$$

$$\theta + 0.693 = \pm 1.567$$

$$\theta = 0.874,\quad -2.26$$

[4 marks]

Answer 7

$$I_n = \int_0^{\frac{\pi}{4}} tan^n x\, dx$$

$$= \int_0^{\frac{\pi}{4}} tan^{n-2} x \; tan^2 x \; dx$$

$$= \int_0^{\frac{\pi}{4}} tan^{n-2} x \left(sec^2 x - 1 \right) dx$$

$$= \int_0^{\frac{\pi}{4}} sec^2 x \; tan^{n-2} x \; dx - \int_0^{\frac{\pi}{4}} tan^{n-2} x \; dx$$

$$= \int_0^{\frac{\pi}{4}} sec^2 x \; tan^{n-2} x \; dx - I_{n-2}$$

$$= \left[\frac{tan^{n-1} x}{n - 1} \right]_0^{\frac{\pi}{4}} - I_{n-2}$$

$$= \left[\frac{1}{n - 1} \right] - I_{n-2} \qquad \square$$

[5 marks]

(b) $$\int_0^{\frac{\pi}{4}} \left(3 + tan^2 x \right)^2 dx = \int_0^{\frac{\pi}{4}} 9 + 6\, tan^2 x + tan^4 x \; dx$$

$$= 9\, I_0 + 6\, I_2 + I_4$$

$$= 9\, I_0 + 6\, I_2 + \left(\frac{1}{4 - 1} \right) - I_2$$

$$= 9\, I_0 + 5\, I_2 + \frac{1}{3}$$

$$= 9\, I_0 + 5 \times \left(\frac{1}{2 - 1} \right) - 5\, I_0 + \frac{1}{3}$$

$$= 4\, I_0 + 5 + \frac{1}{3}$$

$$= 4 \times \frac{\pi}{4} + \frac{16}{3}$$

$$= \pi + \frac{16}{3}$$

[7 marks]

Push The Pace #4

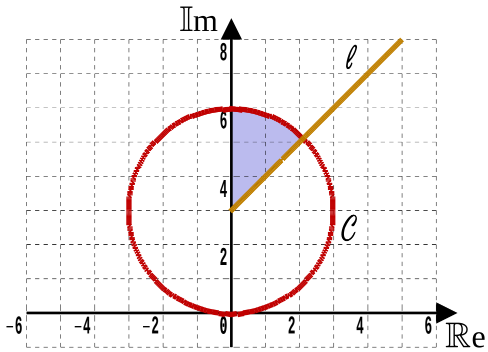
You have thirty-five minutes to answer seven examination questions

Marks Available : 40 (+ 14 bonus)

Further A-Level Pure Mathematics
Push The Pace Revision Papers

Question 1

Further A-Level Examination Question from June 2013, Paper 1, Q6 (OCR)



The Argand diagram shows a half-line ℓ and a circle $\mathcal{C}$.

The circle has centre 3i and passes through the origin.

(i) Write down, in complex number form, the equations of ℓ and $\mathcal{C}$.

[4 marks]

(ii) Write down inequalities that define the region shaded.
(The shaded region includes the boundaries)

[3 marks]

Question 2

Further A-Level Examination Question from June 2019, Paper 1, Q7 (MEI)

A curve has cartesian equation $\left(x^2 + y^2\right)^2 = 2c^2 xy$, where c is a positive constant.

(a) Show that the polar equation of the curve is $r^2 = c^2 \sin 2\theta$

[2 marks]

(b) Sketch the curves $r = c\sqrt{\sin 2\theta}$ and $r = -c\sqrt{\sin 2\theta}$ for $0 \leqslant \theta \leqslant \frac{\pi}{2}$

[3 marks]

(c) Find the area of the region enclosed by one of the loops in part (b).

[3 marks]

Question 3

Advanced Higher Examination Question from May 2019, Q2, (SQA)

Matrix **A** is defined by $\mathbf{A} = \begin{pmatrix} 2 & 1 & 4 \\ -3 & p & 2 \\ -1 & -2 & 5 \end{pmatrix}$ where $p \in \mathbb{R}$

(a) Given that the deteminant of **A** is 3, find the value of p

[3 marks]

Matrix **B** is defined by $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ q & 3 \\ 4 & 0 \end{pmatrix}$ where $q \in \mathbb{R}$

(b) Find **AB**

[2 marks]

(c) Explain why **AB** does not have an inverse.

[1 mark]

Question 4

Further AS-Level Examination Question from May 2019, Q10 (WJEC)

The quadratic equation,

$$px^2 + qx + r = 0$$

has roots α and β, where p, q, r are non-zero constants.

(a) A cubic equation is formed with roots α, β, $\alpha + \beta$

Find the cubic equation with coefficients expressed in terms of p, q, r.

[6 marks]

(b) Another quadratic equation $px^2 - qx - r = 0$ has roots 2α and γ.

Show that $\beta = -2\gamma$

[3 marks]

Question 5

Further A-Level Specimen Examination Question from 2017, Paper 1, Q11 (AQA)

(a) Prove that $\dfrac{\sinh\theta}{1 + \cosh\theta} + \dfrac{1 + \cosh\theta}{\sinh\theta} \equiv 2\coth\theta$

[4 marks]

(b) Solve $\dfrac{\sinh\theta}{1 + \cosh\theta} + \dfrac{1 + \cosh\theta}{\sinh\theta} = 4$

Give your answer in an exact form.

[2 marks]

Question 6

Further A-Level Examination Question from June 2015, Paper FP1, Q3 (WJEC)

The complex number z satisfies the equation,

$$2z - \mathrm{i}\,z^* = \frac{2 + \mathrm{i}}{1 - \mathrm{i}}$$ where z^* denotes the complex conjugate of z

Express z in the form $a + b\mathrm{i}$ where a and b are rational numbers to be found.

[4 marks]

Question 7

Further A-Level Examination Question from June 2010, Paper FP2, Q8 (Edexcel)

(a) Find the value of λ for which $y = \lambda x \sin 5x$ is a particular integral of the differential equation,

$$\frac{d^2y}{dx^2} + 25y = 3\cos 5x$$

[4 bonus marks]

(b) Using your answer to part (a), find the general solution of the differential equation,

$$\frac{d^2y}{dx^2} + 25y = 3\cos 5x$$

[3 Bonus Marks]

Given that at $x = 0$, $y = 0$ and $\dfrac{dy}{dx} = 5$,

(c) find the particular solution of this differential equation, giving your solution in the form $y = f(x)$

[5 bonus marks]

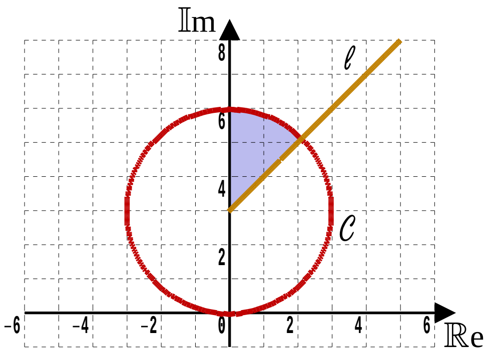
(d) Sketch the curve with equation $y = f(x)$ for $0 \leqslant x \leqslant \pi$

[2 bonus marks]

Push The Pace #4
Answers

Further A-Level Pure Mathematics
Push The Pace Revision Papers

Answer 1



(i) The half-line ℓ has equation $z - 3\text{i} = arg\left(\frac{\pi}{4}\right)$

The circle $\mathcal{C}$ has equation, $|z - 3\text{i}| = 3$

[4 marks]

(ii) The region shaded could be specified by,

$$\left\{ arg\left(\frac{\pi}{4}\right) \leqslant z - 3\text{i} \leqslant arg\left(\frac{\pi}{2}\right) \right\} \cap \{ |z - 3\text{i}| \leqslant 3 \}$$

[3 marks]

Answer 2

(a) Let $x = r\cos\theta$ and $y = r\sin\theta$ in which case $x^2 + y^2 = r^2$

In consequence, $\left(x^2 + y^2\right)^2 = 2c^2xy$ becomes,

$$\left(r^2\right)^2 = 2c^2\,(r\cos\theta)(r\sin\theta)$$

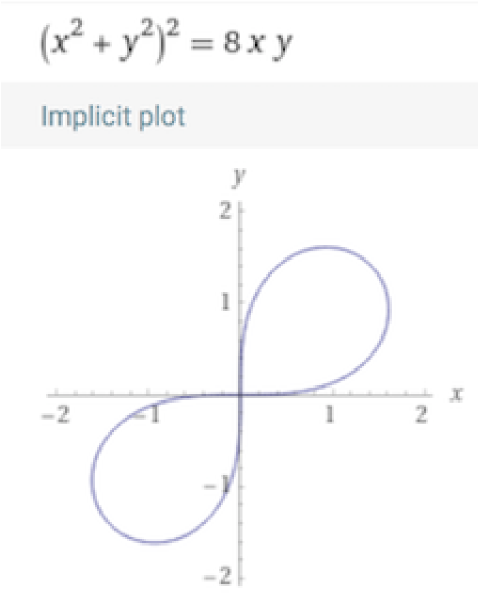
Divide through by r^2

$$r^2 = c^2\left(2\sin\theta\cos\theta\right)$$

$r^2 = c^2\sin 2\theta$ is the polar equation of the curve

[2 marks]

(b) Here is the plot from Wolfram Alpha with $c = 2$



[3 marks]

(c)

$$I = \frac{1}{2}\int r^2\,d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} c^2\sin 2\theta\,d\theta$$

$$= \frac{c^2}{4}\int_0^{\frac{\pi}{2}} 2\sin 2\theta\,d\theta$$

$$= \frac{c^2}{4}\left[-\cos 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{c^2}{4}\times(+2) \qquad \text{and so the area is } \frac{c^2}{2}$$

[3 marks]

Answer 3

(a)

$$|\mathbf{A}| = 3$$

$$\begin{vmatrix} 2 & 1 & 4 \\ -3 & p & 2 \\ -1 & -2 & 5 \end{vmatrix} = 3$$

Expanding along the middle row...

$$3\begin{vmatrix} 1 & 4 \\ -2 & 5 \end{vmatrix} + p\begin{vmatrix} 2 & 4 \\ -1 & 5 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ -1 & -2 \end{vmatrix} = 3$$

$$3 \times 13 + 14p + 6 = 3$$

$$14p = -42$$

$$p = -3$$

[3 marks]

(b) **AB** $= \begin{pmatrix} 2 & 1 & 4 \\ -3 & -3 & 2 \\ -1 & -2 & 5 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ q & 3 \\ 4 & 0 \end{pmatrix}$

$$= \begin{pmatrix} q + 16 & 5 \\ -3q + 8 & -12 \\ -2q + 20 & -7 \end{pmatrix}$$

[2 marks]

(c) **AB** is not a square matrix.

Inverses are only defined for square matrices.

$\therefore$ **AB** has no inverse.

[1 mark]

Answer 4

(a) From the quadratic we get that...

$$p x^2 + qx + r = 0$$

$$x^2 + \frac{q}{p} x + \frac{r}{p} = 0$$

$$\text{Sum: } \alpha + \beta = -\frac{q}{p} \qquad \text{Product: } \alpha\beta = \frac{r}{p}$$

For the new cubic equation we require...

$$\text{Sum: } \alpha + \beta + (\alpha + \beta) = 2(\alpha + \beta)$$

$$= -\frac{2q}{p}$$

$$\text{Sum of Product Pairs: } \alpha\beta + \beta(\alpha + \beta) + (\alpha + \beta) = \alpha\beta + (\alpha + \beta)^2$$

$$= \frac{r}{p} + \frac{q^2}{p^2}$$

$$\text{Product: } \alpha\beta(\alpha + \beta) = -\frac{qr}{p^2}$$

A cubic with roots α, β and $(\alpha + \beta)$ is,

$$x^3 + \left(\frac{2q}{p}\right)x^2 + \left(\frac{r}{p} + \frac{q^2}{p^2}\right)x + \frac{qr}{p^2} = 0$$

[6 marks]

(b)

$$p x^2 - qx - r = 0$$

$$x^2 - \frac{q}{p} x - \frac{r}{p} = 0$$

$$\text{Sum: } 2\alpha + \gamma = \frac{q}{p} \qquad \text{Product: } 2\alpha\gamma = -\frac{r}{p}$$

$$2\alpha\gamma = -\alpha\beta$$

$$\beta = -2\gamma$$

[3 marks]

Answer 5

(a)

$$\begin{aligned}
\text{LHS} &= \frac{sinh\,\theta}{1 + cosh\,\theta} + \frac{1 + cosh\,\theta}{sinh\,\theta} \\
&= \frac{sinh^2\,\theta + (1 + cosh\,\theta)^2}{(1 + cosh\,\theta)\,sinh\,\theta} \\
&= \frac{sinh^2\,\theta + 1 + 2\,cosh\,\theta + cosh^2\,\theta}{(1 + cosh\,\theta)\,sinh\,\theta} \\
&= \frac{sinh^2\,\theta + cosh^2\,\theta - sinh^2\,\theta + 2\,cosh\,\theta + cosh^2\,\theta}{(1 + cosh\,\theta)\,sinh\,\theta} \\
&= \frac{2\,cosh^2\,\theta + 2\,cosh\,\theta}{(1 + cosh\,\theta)\,sinh\,\theta} \\
&= \frac{2\,cosh\,\theta\,(cosh\,\theta + 1)}{(1 + cosh\,\theta)\,sinh\,\theta} \\
&= \frac{2\,cosh\,\theta}{sinh\,\theta} \\
&= 2\,coth\,\theta \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[4 marks]

(b)

$$\begin{aligned}
\frac{sinh\,\theta}{1 + cosh\,\theta} + \frac{1 + cosh\,\theta}{sinh\,\theta} &= 4 \\
2\,coth\,\theta &= 4 \qquad \text{from part (a)} \\
\frac{2}{tanh\,\theta} &= 4 \\
tanh\,\theta &= \frac{1}{2} \\
(\theta &= 0.5493...) \\
\theta &= artanh\left(\frac{1}{2}\right) \\
&= \frac{1}{2} ln\left(\frac{1 + \frac{1}{2}}{1 - \frac{1}{2}}\right) \\
&= \frac{1}{2}\,ln\,3
\end{aligned}$$

[2 marks]

Answer 6

$$2z - \mathrm{i}\,z^{*} = \frac{2 + \mathrm{i}}{1 - \mathrm{i}}$$

$$2(a + b\mathrm{i}) - \mathrm{i}(a - b\mathrm{i}) = \frac{(2 + \mathrm{i})}{(1 - \mathrm{i})} \times \frac{(1 + \mathrm{i})}{(1 + \mathrm{i})}$$

$$2a + 2b\mathrm{i} - a\mathrm{i} + b\,\mathrm{i}^2 = \frac{2 + 3\mathrm{i} + \mathrm{i}^2}{1 - \mathrm{i}^2}$$

$$(2a - b) + (-a + 2b) = \frac{1}{2} + \frac{3}{2}\mathrm{i}$$

$$(4a - 2b) + (-2a + 4b) = 1 + 3\mathrm{i}$$

Equating real parts and equating imaginary parts...

$$\left.\begin{array}{r} 4a - 2b = 1 \\ -2a + 4b = 3 \end{array}\right\} = \left\{\begin{array}{r} 8a - 4b = 2 \\ -2a + 4b = 3 \end{array}\right.$$

$$6x \qquad = 5$$

$$a = \frac{5}{6} \text{ and } b = \frac{7}{6}$$

$$\therefore z = \frac{5}{6} + \frac{7}{6}\mathrm{i}$$

[4 marks]

Answer 7

(a) If $y = \lambda x \sin 5x$ then $\dfrac{dy}{dx} = 5\lambda x \cos 5x + \lambda \sin 5x$

$$\frac{d^2y}{dx^2} = -25\lambda x \sin 5x + 5\lambda \cos 5x + 5\lambda \cos 5x$$

$$\frac{d^2y}{dx^2} + 25y = 3\cos 5x$$

$$-25\lambda x \sin 5x + 5\lambda \cos 5x + 5\lambda \cos 5x + 25(\lambda x \sin 5x) = 3\cos 5x$$

$$10\lambda \cos 5x = 3\cos 5x$$

$$\lambda = \frac{3}{10}$$

[4 Bonus Marks]

(b) Consider $\dfrac{d^2y}{dx^2} + 25y = 0$ which has auxiliary equation $m^2 + 25 = 0$

The complex roots of $\pm 5\mathrm{i}$ lead to the complementary function being,

$$y = A\cos 5x + B\sin 5x$$

$\therefore$ The general solution is $y = A\cos 5x + B\sin 5x + \dfrac{3}{10}x\sin 5x$

[3 Bonus Marks]

(c) Using the initial condition that $x = 0, \; y = 0$ gives,

$$0 = A$$

What's left of the general solution for these initial conditions is,

$$y = B \sin 5x + \frac{3}{10} x \sin 5x$$

$$\frac{dy}{dx} = 5B \cos 5x + \frac{3 \times 5}{10} x \cos 5x + \frac{3}{10} \sin 5x$$

$$= \frac{3}{10} \sin 5x + \left(5B + \frac{3}{2} x\right) \cos 5x$$

Using the initial condition that $x = 0, \; \dfrac{dy}{dx} = 5$ gives,

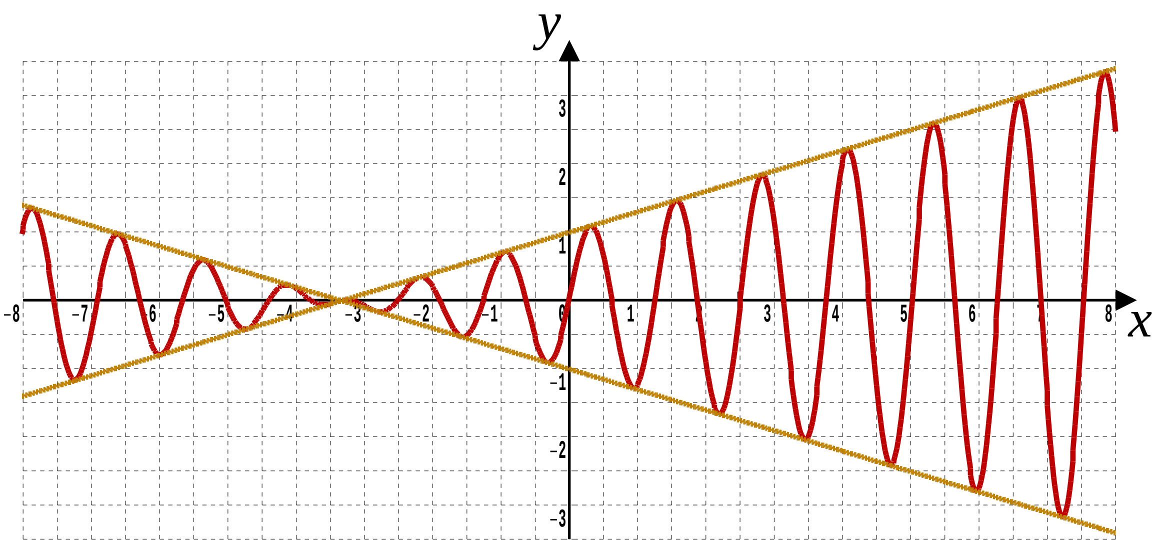
$$5 = 5B \qquad \text{so } B = 1$$

The particular solution is $y = \sin 5x + \dfrac{3}{10} x \sin 5x$

$$= \left(1 + \frac{3}{10} x\right) \sin 5x$$

[5 bonus marks]

(d) This is a sine wave $\left(\text{with period } \dfrac{2\pi}{5}\right)$ with envelope $y = \dfrac{3}{10} x + 1$



[2 bonus marks]