2.5 Answers

2.5.1 Solution to 2.3 Example (Teaching Video)

Prove $G = \{1, 2, 3, 4\}$ under multiplication modulo 5 is a group.

$\times_5$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

$\times_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

Identity: For this group, 1 is the identity element, the element that "does nothing" under the binary operation of multiplication modulo 5.

Closure : As the table shows all possible multiplications it can be seen that no number can occur as an answer that's not an element of $G = \{1, 2, 3, 4\}$.

Inverse : Each element has a unique inverse. 2 and 3 are each the other's inverse and 4 is it's own inverse as is 1.

$2 \times 3 \equiv 3 \times 2 \equiv 1 \pmod 5 \quad 4 \times 4 \equiv 1 \pmod 5$

Associativity : This is a property inherited from of multiplication of integers.

2.5.2 Solutions to 2.4 Exercise

Answer 1

(i)

$\times_{18}$	1	5	7	11	13	17
1	1	5	7	11	13	17
5	5	7	17	1	11	13
7	7	17	13	5	1	11
11	11	1	5	13	17	7
13	13	11	1	17	7	5
17	17	13	11	7	5	1

[2 marks]

(ii) The identity element is 1

[1 mark]

(iii) The inverse of 5 is 11 because $5 \times 11 = 1 \pmod{18}$

[1 mark]

(iv) The inverse of 17 is 17 because $17 \times 17 = 1 \pmod{18}$

[1 mark]

Answer 2

$x * y = x^2 y$

(i) $5 * 6 = 5^2 \times 6 = 150$

(ii) $6 * 5 = 6^2 \times 5 = 180$

[2 marks]

Answer 3

(i)

$\times_{12}$	1	3	7	9
1	1	3	7	9
3	3	9	9	3
7	7	9	1	3
9	9	3	3	9

[2 marks]

(ii) Yes, the closure property is satisfied.

[1 mark]

(iii)
- The Latin square property does not hold.
- Two of the elements, the 3 and the 9, do not have an inverse.

[1 mark]

Answer 4

$a * b = ab + 1$

(a) **(i)** $4 * 7 = 4 \times 7 + 1$

$= 29$

(ii) $(-3) * 5 = (-3) \times 5 + 1$

$= -14$

[2 marks]

(b) $2 * (3 * 4) = 2 * (3 \times 4 + 1)$

$= 2 * 13$

$= 2 \times 13 + 1$

$= 27$

$(2 * 3) * 4 = (2 \times 3 + 1) * 4$

$= 7 * 4$

$= 7 \times 4 + 1$

$= 29$

Thus $2 * (3 * 4) \neq (2 * 3) * 4$

$\therefore$ $*$ is not associative □

[2 marks]

Answer 5

(i)

$\circ$	**0**	**1**	**2**	**4**	**5**	**6**
0	**0**	**1**	**2**	**4**	**5**	**6**
1	**1**	**4**	**0**	**6**	**2**	**5**
2	**2**	**0**	**5**	**1**	**6**	**4**
4	**4**	**6**	**1**	**5**	**0**	**2**
5	**5**	**2**	**6**	**0**	**4**	**1**
6	**6**	**5**	**4**	**2**	**1**	**0**

[3 marks]

(ii) The identity element is 0

[1 mark]

(iii) The inverse of 5 is 4

[1 mark]

(iv) The inverse of 1 is 2

[1 mark]

(v) Other than 0, the self-inverse element is 6

[1 mark]

Answer 6

Method 1 :

$$\begin{aligned} x \circ y &= x^2 y \\ odd \circ odd &= odd^2 \times odd \\ &= odd \times odd \\ &= odd \end{aligned}$$

So, when any two elements from G are combined the result

is an element from G which is what is means to be closed ☐

Method 2 :

Let the two odd numbers be $2n + 1$ and $2m + 1$

$$\begin{aligned} (2n + 1) \circ (2m + 1) &= (2n + 1)^2 (2m + 1) \\ &= 2\left(4mn^2 + 4nm + m + 2n^2 + 2n\right) + 1 \\ &= 2k + 1, \text{ another odd number} \end{aligned}$$

$\therefore$ G is closed under $\circ$

[2 marks]

Answer 7

(a) From the Cayley table provided,

$\otimes$	**1**	**2**	**3**	**4**	**5**
1	**1**	**2**	**3**	**4**	**5**
2	**2**	**1**	**5**	**3**	**4**
3	**3**	**4**	**2**	**5**	**1**
4	**4**	**5**	**1**	**2**	**3**
5	**5**	**3**	**4**	**1**	**2**

(i) $2 \otimes (5 \otimes 3) = 2 \otimes 4$

$= 3$

(ii) $(2 \otimes 5) \otimes 3 = 4 \otimes 3$

$= 1$

[2 marks]

(b) The table does not have associativity and so is not of a group.

[1 mark]

Answer 8

$x \circ y = xy + x$

Let's check to see if associativity holds.

In other words, for any three elements of G, say a, b and c, is it true that,

$a \circ (b \circ c) = (a \circ b) \circ c$?

$$\begin{aligned} \text{LHS} &= a \circ (b \circ c) \\ &= a \circ (bc + b) \\ &= a(bc + b) + a \\ &= abc + ab + a \end{aligned} \qquad \begin{aligned} \text{RHS} &= (a \circ b) \circ c \\ &= (ab + a) \circ c \\ &= (ab + a)c + (ab + a) \\ &= abc + ac + ab + a \end{aligned}$$

As LHS $\neq$ RHS the proposed group lacks associativity and so is not a group under the binary operation $\circ$.

[4 marks]

Answer 9

Assume there exists a column in a Cayley table in which the same element occurs twice. Given two distinct elements, *p* and *q*, in column *b* the situation is;

$\circ$	…	b	…
…	…	…	…
p	…	$p\circ b$	…
…	…	…	…
q	…	$q\circ b$	…
…	…	…	…

Under the assumption that the same element occurs twice,

$$p \circ b = q \circ b$$

$$(p \circ b) \circ b^{-1} = (q \circ b) \circ b^{-1} \qquad \text{Every element in a group has an inverse}$$

$$p \circ (b \circ b^{-1}) = q \circ (b \circ b^{-1}) \qquad \text{A group has associativity}$$

$$p \circ e = q \circ e \qquad e \text{ is the identity element}$$

$$p = q$$

But this contradicts the starting assumption that *p* and *q* are distinct elements.

$\therefore$ the same element cannot occur twice in any column in a group's Cayley table.

The two parts together prove that all group Cayley tables will be Latin squares.

[2 marks]

Answer 10

(a)

$$
\begin{aligned}
ff(x) &= f\left(\frac{3x-1}{7x-2}\right) \\
&= \frac{3\left(\frac{3x-1}{7x-2}\right)-1}{7\left(\frac{3x-1}{7x-2}\right)-2} \\
&= \frac{9x-3-7x+2}{21x-7-14x+4} \\
&= \frac{2x-1}{7x-3} \quad \square
\end{aligned}
$$

[3 marks]

(b)

$$
\begin{aligned}
fff(x) &= ff\left(\frac{3x-1}{7x-2}\right) \\
&= f\left(\frac{2x-1}{7x-3}\right) \\
&= \frac{3\left(\frac{2x-1}{7x-3}\right)-1}{7\left(\frac{2x-1}{7x-3}\right)-2} \\
&= \frac{6x-3-7x+3}{14x-7-14x+6} \\
&= \frac{-x}{-1} \\
&= x \quad \square
\end{aligned}
$$

[3 marks]

(c) **(i)** The identity element is $fff(x)$ because $fff(x) = x$

[1 mark]

(ii) $fff(x) \circ ff(x) = e \circ ff(x)$ $\quad$ e is the identity

$$= ff(x)$$

[1 mark]

(iii) The inverse of $ff(x)$ is $f(x)$ because,

$$ff(x) \circ f(x) = f(x) \circ ff(x) = e$$

[1 mark]