5.8 Answers to 5.7 Exercise

Answer 1

(i)

$\circ$	***A***	***B***	***C***	***D***
A	***D***	***C***	***B***	***A***
B	***C***	***D***	***A***	***B***
C	***B***	***A***	***D***	***C***
D	***A***	***B***	***C***	***D***

[3 marks]

(ii) D is the identity element.

[1 mark]

(iii) $A^2 = B^2 = C^2 = D$ (the identity element)

So A, B and C are of order 2, with D having order 1

[2 marks]

(iv) The group of mattress transformations is not cyclic because none of the elements was of order 4.

That is, none of the elements will generate the whole group.

[1 mark]

(v) The mattress transformations group is isomorphic to K_4 as that is the only fundamental group of order 4 that is not cyclic.

[1 mark]

Answer 2

(i) m is the identity element

Look for the yellow row that is the same as the red row and the green element at the start of that yellow row is the identity element.

[1 mark]

(ii) Every element is its own inverse !

The quick way to spot this from the diagonal on the Cayley table.

Every element on that diagonal is, m the identity element.

Here is a short proof that s is its own inverse;

$$s * s = m$$

$$s^{-1} * (s * s) = s^{-1} * m \quad \text{Pre multiply both sides by } s^{-1}$$

$$(s^{-1} * s) * s = s^{-1} \quad \text{Associativity}$$

$$m * s = s^{-1}$$

$$s = s^{-1}$$

[2 marks]

(iii) The table is not of a Cyclic group because all of the elements are of order 2 except the identity which is of order 1.

No element is of order 8. That is, no element generates the whole group.

$\therefore$ The group is not cyclic.

[2 marks]

(iv) From the Catalogue of known groups, the only group that has order 8 and all elements of order 2 is the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and so the unknown group must be isomorphic to this.

[2 marks]

Answer 3

(i)

$$\text{LHS} = s * t$$

$$= \begin{pmatrix} G & O & P \\ O & G & P \end{pmatrix} * \begin{pmatrix} G & O & P \\ O & P & G \end{pmatrix}$$

$$= \begin{pmatrix} G & O & P \\ G & P & O \end{pmatrix}$$

$$= r$$

$$= \text{RHS} \qquad \square$$

[2 marks]

(ii)

$$t * s = \begin{pmatrix} G & O & P \\ O & P & G \end{pmatrix} * \begin{pmatrix} G & O & P \\ O & G & P \end{pmatrix}$$

$$= \begin{pmatrix} G & O & P \\ P & O & G \end{pmatrix}$$

$$= u$$

$$\neq s * t \qquad \square$$

[2 marks]

(iii)

$$s^{-1} = \begin{pmatrix} G & O & P \\ O & G & P \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} G & O & P \\ O & G & P \end{pmatrix}$$

$= s$ It's self inverse

[2 marks]

(iv)

*	e	r	s	t	u	v
e	e	r	s	t	u	v
r	r	e	v	u	t	s
s	s	t	e	r	v	u
t	t	s	u	v	r	e
u	u	v	t	s	e	r
v	v	u	r	e	s	t

[4 marks]

(v) Closure : From the Cayley table it is observed that all possible combinations of two elements from the set $\{e, r, s, t, u, v\}$ yield only elements already in that set.

$\therefore \ (G, *)$ is closed

Identity : The set has a unique identity, e

Inverses : The inverse of e is e

The inverse of r is r

The inverse of s is s

The inverse of t is v

The inverse of u is u

The inverse of v is t

$\therefore$ Each element of G has a unique inverse

With the assumption that associativity holds, the axioms for G to be a group have been shown to be satisfied.

[3 marks]

(vi) This question has looked at the set of permutations of three objects which The Catalogue of all Possible Groups calls S_3

Other answers are possible such as showing the group is not cyclic.

[2 marks]

Answer 4

$\times_{30}$	1	7	11	13	17	19	23	29
1	1	7	11	13	17	19	23	29
7	7	19	17	1	29	13	11	23
11	11	17	1	23	7	29	13	19
13	13	1	23	19	11	7	29	17
17	17	29	7	11	19	23	1	13
19	19	13	29	7	23	1	17	11
23	23	11	13	29	1	17	19	7
29	29	23	19	17	13	11	7	1

(i) $\langle 1 \rangle = \{1\}$ $\quad |1| = 1$

$\langle 7 \rangle = \{1, 7, 13, 19\}$ $\quad |7| = 4$

$\langle 11 \rangle = \{1, 11\}$ $\quad |11| = 2$

$\langle 13 \rangle = \{1, 7, 13, 19\}$ $\quad |13| = 4$

$\langle 17 \rangle = \{1, 17, 19, 23\}$ $\quad |17| = 4$

$\langle 19 \rangle = \{1, 19\}$ $\quad |19| = 2$

$\langle 23 \rangle = \{1, 17, 19, 23\}$ $\quad |23| = 4$

$\langle 29 \rangle = \{1, 29\}$ $\quad |29| = 2$

[4 marks]

(ii) The part (i) answer found two cyclic subgroups of order 4.
They are, {1, 7, 13, 19} or {1, 17, 19, 23}

[2 marks]

(iii) The Klein four-group, K_4 is not cyclic.
One idea is to start with a cyclic subgroup of order 2, and add an
from one of the other subgroups of order 2 that's not is a cyclic subgroup.
{1, 11, __, 29} $\Rightarrow$ {1, 11, 19, 29} which is isomorphic to K_4

[2 marks]

(iv) H is isomorphic to $\mathbb{Z}_8$ which is cyclic (from the Catalogue)
$\therefore$ As G in not cyclic it is not isomorphic to H

[2 marks]