Shrewsbury School
The Arnold Hagger Prize 2013

Warning!

If you are using a calculator to answer these questions you are approaching them in the wrong way! None of these questions have any marks at all for the answer. All the marks are for how you explain your conclusions.

Expect to spend a lot of time doodling, some time thinking, and a small amount of time writing your thoughts out in coherent mathematics. Your marker is a bear of little brain, and needs things explaining slowly to get it!

Don't aim to attempt all the questions. Part of the skill is picking questions you can do. A small number of complete solutions will score more highly than a large number of partial ones.

The questions are very roughly in order of difficulty.

- *The first 7 questions are worth 5 marks each*
- *Q8 is worth 8 marks*
- *Q9 onwards are worth 10 each*
- *Try for few complete, not lots partial.*

You have 1 hour 30 minutes. Good luck!

Sketch solutions will be posted sometime...

Do or do not. There is no Try
Yoda, The Empire Strikes Back

Mango Turnip Fecit MMXIII

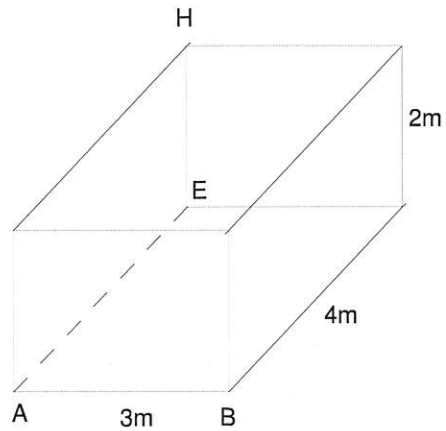
1) I drive 10 miles at 10 miles per hour, followed by 10 miles at 5 miles per hour. What is my average speed?
2) ABC is an equilateral triangle of side 2 units. 5 dots are placed at random inside the triangle. Convince me that there are two dots which are less than 1 unit apart.

3) a) Find the last digit of 17^{10}

b) Find the last digit of $3^{122}\ 5^{600}\ 112^{450}$

4) Sketch the graph of $(x + y - 1)(x^2 + y^2 - 10) = 0$

5) A room is cuboidal, of width 3m, length 4m, height 2m. A fly starts to make its way from B to H, crawling along the floor, then up the wall. The fly is intelligent, and always takes the shortest route. X is the point where the fly crosses AE

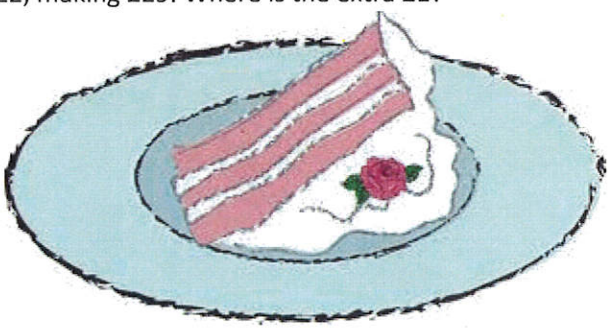


If it travels with equal speed on floor and wall, find the distance AX

6) Albert, Brian and Christine go out for a meal, which costs £25. They misread the bill thinking it says £27. They give the waiter £30 (£10 each) and ask for £3 change. The waiter puts £25 in the till, gives each of the friends £1 and pockets the extra £2

After they have gone, the waiter does some calculations. The friends paid £(10-1) = £9 each. Three friends = total of £27. The waiter kept £2, making £29. Where is the extra £1?

7) A cake is cylindrical in shape, with radius r and height h. It is iced on the side and top only. Slices are removed from the cake in the traditional way, viewed from the top as sectors. What dimensions will ensure that the top of the slice of the slice has the same area of icing as the side of the slice?



8) Prove that $\sqrt[7]{7!} < \sqrt[8]{8!}$ (7! = 7 x 6 x 5 x 4 x ... x 1)

9) Solve the equation

$$\cfrac{5}{3+\cfrac{x}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cdots}}}}}}} = 10$$

(The pattern established by the dots continues for ever)

10) the bill in Q6 remains unsettled, so Albert (A), Brian(B), and Christine(C) are to fight a three cornered pistol duel. All know that A's chance of hitting his target is 0.3, C's is 0.5 and B never misses. They are to fire at their choice of target in succession in order A, B, C, cyclically (but a hit man is out of the contest completely) until only one man remains. What should A's strategy be?

11) Find solutions to the equation 51x+21y=18 where x and y are both whole numbers

(i) How many solutions are there?

(ii) Can you find solutions to 51x-21y=18

12) Consider the following sequence of numbers:
0, 1, 0.5, 0.75, 0.625, ..., x_{n-2}, x_{n-1} , $x_{n,...}$

(i) Express x_n in terms of x_{n-1}, x_{n-2}.

(ii) Prove that $x_n = \frac{2^{n-1}+(-1)^n}{3\times 2^{n-2}}$ for all natural numbers n

13)If, in a large integer, the difference between the sum of all its digits in even positions and the sum of all digits in odd positions is zero or a multiple of 11, then the number is a multiple of 11. True or False? Give a proof or a clear illustration of falsehood.

14) Prove that $x^3-6x^2+11x-6$ is divisible by 3 if x is any whole number

15) Which is more likely – getting 1 six when 6 dice are rolled, 2 sixes when 12 dice are rolled, or 3 sixes when 18 dice are rolled?

----------BONUS----------

Only go below here if you have cracked all the above questions

16)Show that $x^3 - 3xbc + b^3 + c^3 \equiv (x + b + c)Q(x)$ where Q(x) is a quadratic polynomial
Show that 2Q(x) is the sum of three perfect squares.

The equations $ay^2 + by + c = 0$ and $by^2 + cy + a = 0$ have a common root, k. The coefficients a,b are nonzero, and $ac \neq b^2$. Show that

$$(ac - b^2)k = bc - a^2$$

And discover similar expression involving k^2. Hence show that

$$(ac - b^2)(ab - c^2) = (bc - a^2)^2$$

And that

$$a^3 - 3abc + b^3 + c^3 = 0$$