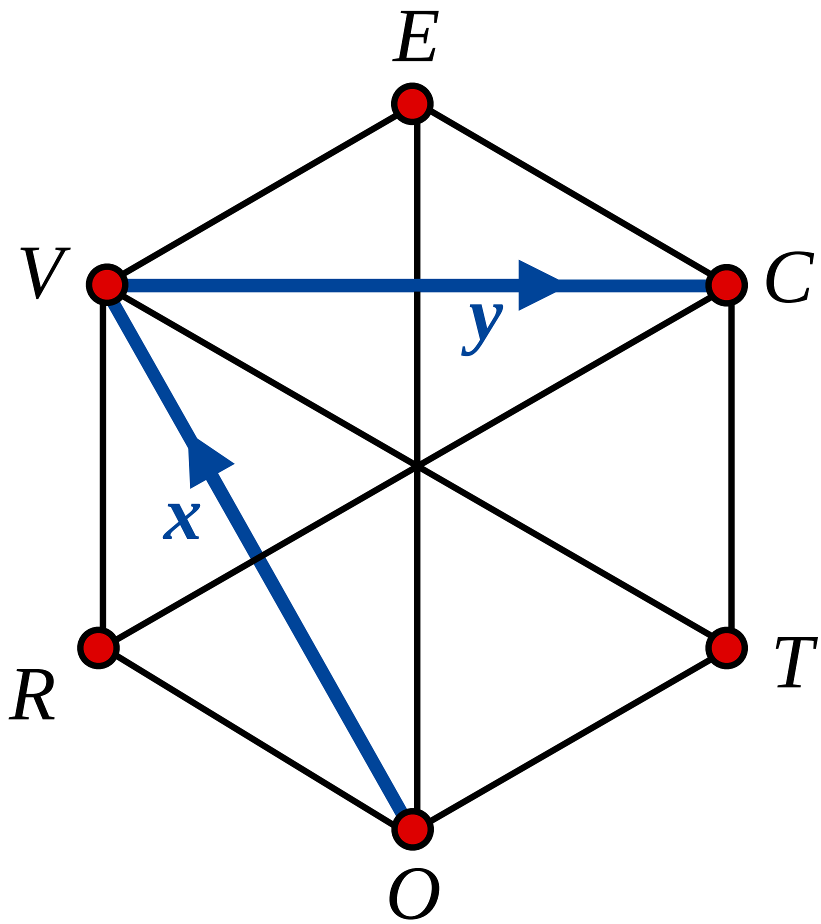
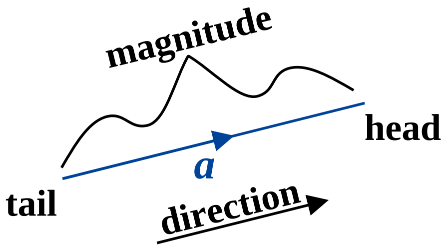
Vectors



Arrows on a mission

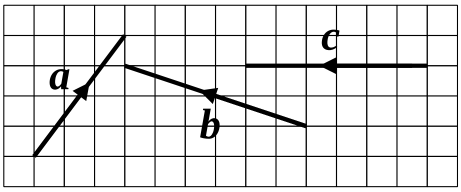
1.1 Arrows on paper

A vector can be thought of as being an arrow on the page.
The arrow has a fixed length, called its magnitude, and acts in a fixed direction.
Its location, however is not fixed. It is free to be **TRANSLATED** about the page.



The direction of a vector is from its tail to its head and its magnitude is the length between the tail and the head.

1.1.1 Example



Vector $\boldsymbol{a}$ is described mathematically by using the fact that to pass from its tail to its head one journeys 3 squares across and 4 squares up.

Thus: Alternatively:

$$\boldsymbol{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \qquad\qquad \boldsymbol{a} = 3\boldsymbol{i} + 4\boldsymbol{j}$$

In the alternative description of vector $\boldsymbol{a}$ the $\boldsymbol{i}$ and the $\boldsymbol{j}$ are vectors of magnitude 1.
Vectors with a magnitude of 1 are called unit vectors.
$\boldsymbol{i}$ is a unit vector in the positive x-axis direction.
$\boldsymbol{j}$ is a unit vector in the positive y-axis direction.
So even if vector $\boldsymbol{a}$ is written as $\boldsymbol{a} = 4\boldsymbol{j} + 3\boldsymbol{i}$, it would still be "3 across and 4 up".
When written as a column vector, however, swapping the 3 and the 4 is disastrous as the upper number is always the horizontal x-axis instruction and the lower number is always the vertical y-axis instruction.

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} \neq \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

1.1.2 Exercise

Write down two mathematical descriptions for each of the vectors $\boldsymbol{b}$ and $\boldsymbol{c}$.

[4 marks]

1.2 Factorising a vector

It is often helpful to pull out any common integer term from a vector.

1.2.1 Example

To factorise vector ***g*** where $\boldsymbol{g} = \begin{pmatrix} 25 \\ 15 \end{pmatrix}$

notice that both 25 and 15 divide by 5. $\therefore\ \boldsymbol{g} = 5\begin{pmatrix} 5 \\ 3 \end{pmatrix}$

1.2.2 Exercise

Factorise out any common term from each of these vectors.

$$\boldsymbol{m} = \begin{pmatrix} 20 \\ 55 \end{pmatrix} \qquad \boldsymbol{n} = 12\,\boldsymbol{i} + 8\,\boldsymbol{j}$$

[2 marks]

1.3 Parallel Vectors

To show that two vectors are parallel, show that they have the same direction.

1.3.1 Example

$$\boldsymbol{s} = \begin{pmatrix} 16 \\ 20 \end{pmatrix} \qquad \boldsymbol{t} = \begin{pmatrix} 28 \\ 35 \end{pmatrix}$$

To show that vectors ***s*** and ***t*** are parallel, first, factorise out the common factors,

$$\boldsymbol{s} = 4\begin{pmatrix} 4 \\ 5 \end{pmatrix} \qquad \boldsymbol{t} = 7\begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

It's now obvious that they have the same direction !
One vector has simply got more magnitude (length of the vector) than the other, but in the same direction.

1.3.2 Exercise

Show that the following vectors are parallel,

$$\boldsymbol{a} = \begin{pmatrix} 12 \\ 18 \end{pmatrix} \qquad \boldsymbol{z} = \begin{pmatrix} 16 \\ 24 \end{pmatrix}$$

[3 marks]

1.4 Vector Arithmetic

Vectors can be added to, or subtracted from, each other.
They can also be multiplied by a number.

1.4.1 Example

Vectors $\boldsymbol{a}$, $\boldsymbol{c}$ and $\boldsymbol{e}$ are : $\boldsymbol{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ $\quad \boldsymbol{c} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$ $\quad \boldsymbol{e} = \begin{pmatrix} -9 \\ 3 \end{pmatrix}$

Here is the calculation to work out the vector $2\,\boldsymbol{a} + \boldsymbol{c} - \boldsymbol{e}$;

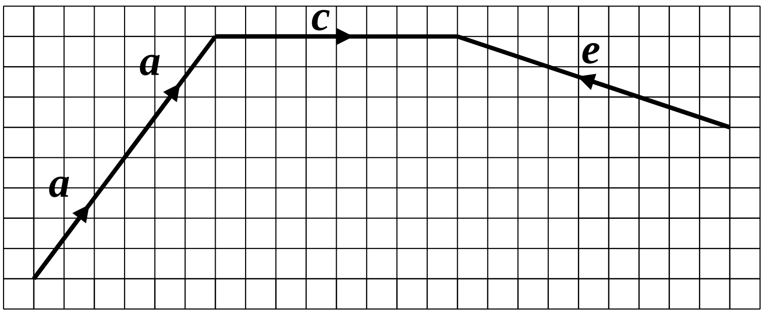
$$2\,\boldsymbol{a} + \boldsymbol{c} - \boldsymbol{e} = 2 \times \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 8 \\ 0 \end{pmatrix} - \begin{pmatrix} -9 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \times 3 + 8 - (-9) \\ 2 \times 4 + 0 - 3 \end{pmatrix}$$

$$= \begin{pmatrix} 23 \\ 5 \end{pmatrix}$$

The calculation can be explained using geometry.
Addition corresponds to joining vectors up “head to tail”.
Subtraction corresponds to joining vectors up “head to head”, (or “tail to tail).
So the above calculation corresponds to the following diagram.

On the diagram, draw in the vector representing the answer.



[1 mark]

1.4.2 Exercise

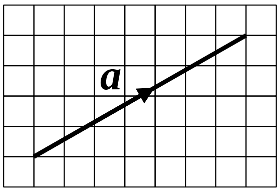
If vectors $\boldsymbol{a}$, $\boldsymbol{c}$ and $\boldsymbol{e}$ are once again,

$$\boldsymbol{a} = 3\,\boldsymbol{i} + 4\,\boldsymbol{j} \qquad \boldsymbol{c} = 8\,\boldsymbol{i} \qquad \boldsymbol{e} = -9\,\boldsymbol{i} + 3\,\boldsymbol{j}$$

Calculate the vector $\boldsymbol{a} + 2\,\boldsymbol{c} - \boldsymbol{e}$

[3 marks]

1.5 The Magnitude of a vector



A vector can be used to represent many aspects of the physical world: a force, a velocity or an acceleration, for example. So it's confusing to talk about a vector's ***length*** because forces, velocities and accelerations are not lengths.
A new word is needed for the length of a vector.
The word is **MAGNITUDE**.

1.5.1 Example

$$\boldsymbol{a} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

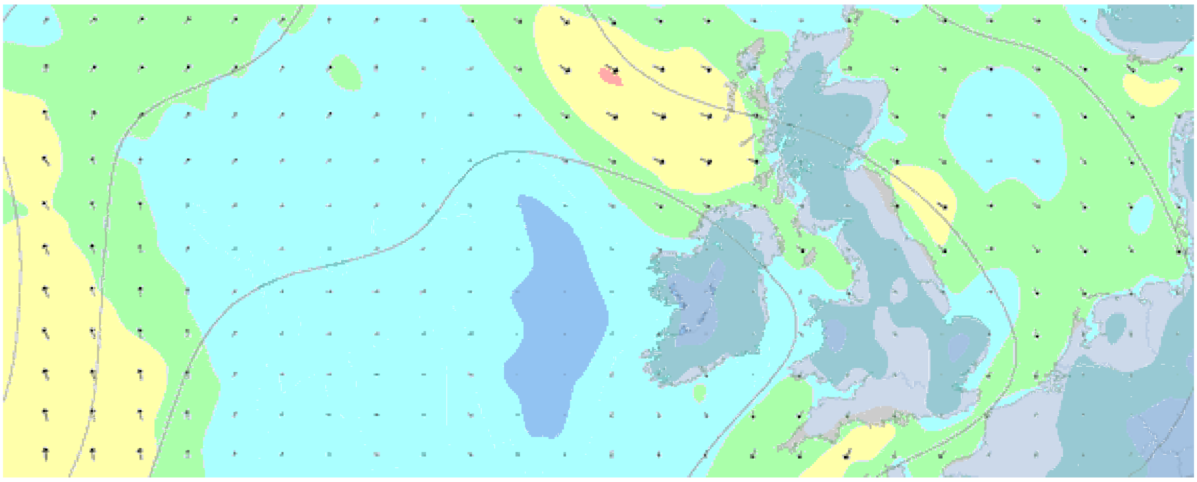
$$\begin{aligned} |\boldsymbol{a}| &= \sqrt{7^2 + 4^2} \\ &= \sqrt{49 + 16} \\ &= \sqrt{65} \end{aligned}$$

$|\boldsymbol{a}|$ is read as “the magnitude of vector ***a***”

1.5.2 Exercise

Find $|\boldsymbol{p}|$ giving the answer is surd form where $\boldsymbol{p} = \begin{pmatrix} 6 \\ 11 \end{pmatrix}$

[2 marks]

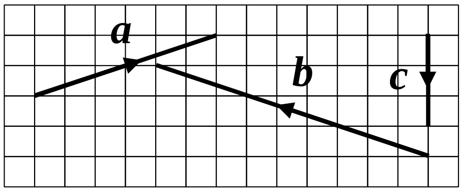


Vectors in use on a weather chart.
Each vector shows the direction of the wind at a point and the magnitude
of the vector corresponds to the strength of the wind at that point.
These vectors are tied to a location and are not free to be translated.
Such a collection of fixed vectors is termed a vector field.

1.6 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable.
Make the method used clear.
Marks available : 50

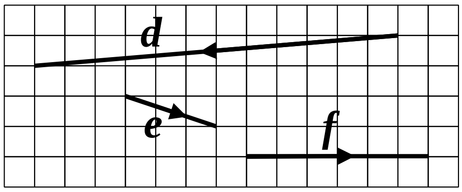
Question 1



Write the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ in the form $\begin{pmatrix} p \\ q \end{pmatrix}$ where p and q are integers.

[3 marks]

Question 2



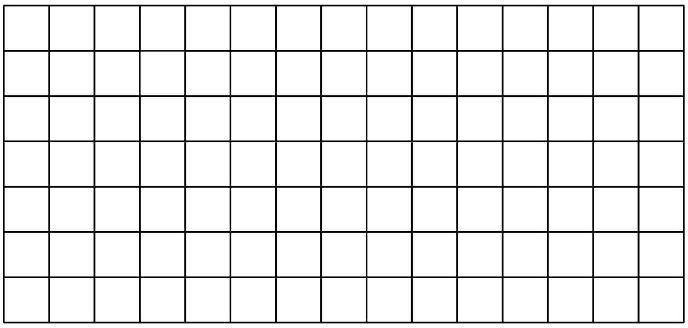
Write the vectors $\boldsymbol{d}$, $\boldsymbol{e}$ and $\boldsymbol{f}$ in the form $p\,\mathbf{i} + q\,\boldsymbol{j}$ where p and q are integers.

[3 marks]

Question 3

On the grid draw the following vectors, labelling each with its letter and an arrow.

$$\boldsymbol{a} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

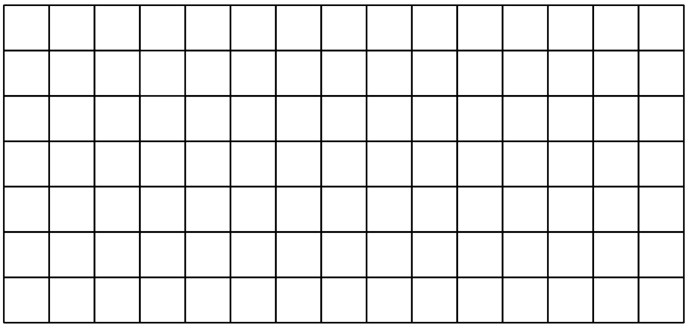


[2 marks]

Question 4

On the grid draw the following vectors, labelling each with its letter and an arrow.

$$\boldsymbol{c} = 2\boldsymbol{i} - 4\boldsymbol{j} \qquad \boldsymbol{d} = -2\boldsymbol{i} + 5\boldsymbol{j}$$



[2 marks]

Question 5

Factorise completely the vectors $\boldsymbol{p}$ and $\boldsymbol{q}$.

$$\boldsymbol{p} = \begin{pmatrix} 22 \\ 77 \end{pmatrix} \qquad \boldsymbol{q} = \begin{pmatrix} 20 \\ 24 \end{pmatrix}$$

[2 marks]

Question 6

Show that vectors ***y*** and ***z*** are parallel.

$$\boldsymbol{y} = \begin{pmatrix} 15 \\ 12 \end{pmatrix} \qquad \boldsymbol{z} = \begin{pmatrix} 35 \\ 28 \end{pmatrix}$$

[2 marks]

Question 7

Show that the following vectors are parallel,

$$\boldsymbol{c} = 12\,\boldsymbol{i} + 15\boldsymbol{j} \qquad \boldsymbol{v} = 28\,\boldsymbol{i} + 35j$$

[2 marks]

Question 8

$$\boldsymbol{T} = \begin{pmatrix} 8 \\ 9 \end{pmatrix}$$

Find $|\boldsymbol{T}|$ giving the answer in surd form.

[2 marks]

Question 9

$$\boldsymbol{m} = \begin{pmatrix} -7 \\ 5 \end{pmatrix}$$

Find $|\boldsymbol{m}|$ giving the answer in surd form.

[2 marks]

Question 10

Let vectors $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ be :

$$\boldsymbol{A} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \qquad \boldsymbol{C} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

Find :

(i) $\boldsymbol{A} + \boldsymbol{B} + \boldsymbol{C}$

[1 mark]

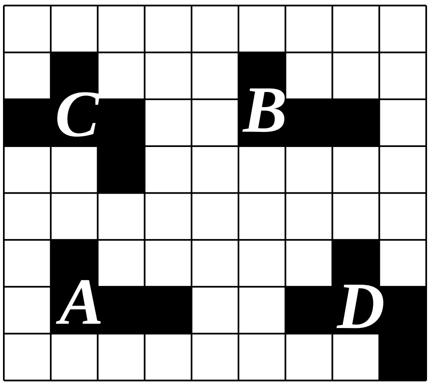
(ii) $2\,\boldsymbol{A} + \boldsymbol{B} + 3\,\boldsymbol{C}$

[2 marks]

(iii) $\boldsymbol{A} + 2\,\boldsymbol{B} + 3\,\boldsymbol{C}$

[2 marks]

Question 11



(i) **(a)** Write down the vector that translates shape *A* onto shape *B*.

[1 mark]

(b) Circle which describes the direction of your part **(i) (a)** answer.

$\begin{pmatrix} -1 \\ -1 \end{pmatrix}$ $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

[1 mark]

(c) Circle which describes the direction of the vector that translates shape *B* onto shape *A*.

$\begin{pmatrix} -1 \\ -1 \end{pmatrix}$ $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

[1 marks]

(ii) **(a)** Write down the vector that translates shape *C* onto shape *D*.

[1 mark]

(b) Circle which describes the direction of your part **(ii) (a)** answer.

$\begin{pmatrix} -3 \\ -2 \end{pmatrix}$ $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$

[1 mark]

(c) Circle which describes the direction of the vector that translates shape *D* onto shape *C*.

$\begin{pmatrix} -3 \\ -2 \end{pmatrix}$ $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$

[1 mark]

Question 12

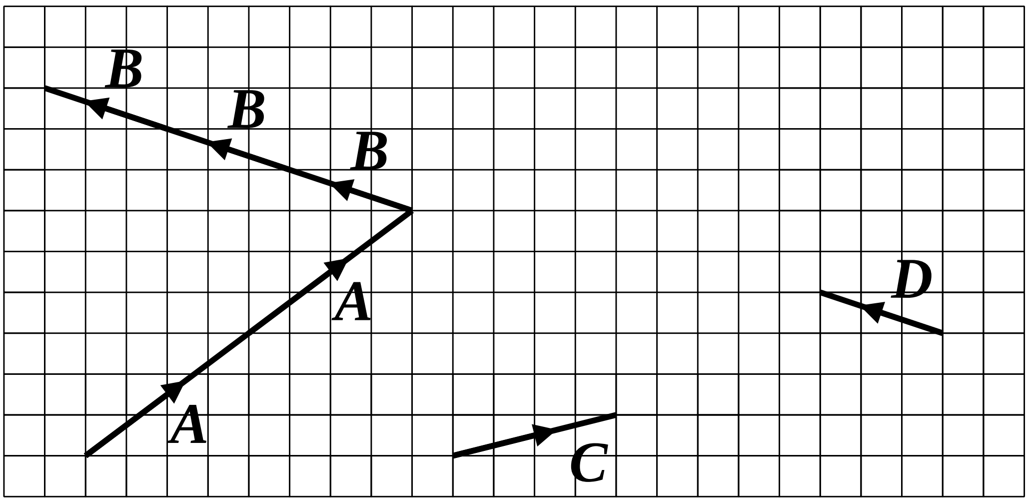
Let vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ be :

$$\boldsymbol{A} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

(i) Calculate vector $\boldsymbol{R}$ where $\boldsymbol{R} = 2\,\boldsymbol{A} + 3\,\boldsymbol{B}$

[2 marks]

(ii) On the diagram below, draw vector $\boldsymbol{R}$ in the appropriate place.



[1 mark]

Question 13

Let vectors $\boldsymbol{C}$ and $\boldsymbol{D}$ be :

$$\boldsymbol{C} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \qquad \boldsymbol{D} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

(i) Calculate vector $\boldsymbol{S}$ where $\boldsymbol{S} = 3\,\boldsymbol{C} + 2\,\boldsymbol{D}$

[2 marks]

(ii) Complete the diagram above to illustrate the vector sum $\boldsymbol{S} = 3\,\boldsymbol{C} + 2\,\boldsymbol{D}$

[1 mark]

Question 14

Let vectors $\boldsymbol{D}$, $\boldsymbol{E}$ and $\boldsymbol{F}$ be :

$$\boldsymbol{D} = -6\boldsymbol{i} - 2\boldsymbol{j} \qquad \boldsymbol{E} = 4\boldsymbol{i} - 3\boldsymbol{j} \qquad \boldsymbol{F} = -5\boldsymbol{i} + 3\boldsymbol{j}$$

Find :

(i) $2\boldsymbol{D} + 2\boldsymbol{E} + 3\boldsymbol{F}$

[2 marks]

(ii) $\boldsymbol{D} + 5\boldsymbol{E} + 2\boldsymbol{F}$

[2 marks]

(iii) $3(\boldsymbol{E} + 2\boldsymbol{F}) - \boldsymbol{D}$

[2 marks]

Question 15

(a) Two translations are described by the following vectors;

$$F = \begin{pmatrix} 7 \\ -2 \end{pmatrix} \qquad G = \begin{pmatrix} -3 \\ 10 \end{pmatrix}$$

Describe the single transformation which is equivalent to combining the two translations.

[1 mark]

(b) Two translations are described by the following vectors;

$$\boldsymbol{f} = 7\,\boldsymbol{a} - 2\,\boldsymbol{b} \qquad \boldsymbol{g} = -3\,\boldsymbol{a} + 10\,\boldsymbol{b}$$

Describe the single transformation which is equivalent to combining the two translations.
(You will have ***a*** and ***b*** in your answer)

[1 mark]

Question 16

Show that the following vectors are parallel,

$$\boldsymbol{b} = \begin{pmatrix} 45 \\ 15 \end{pmatrix} \qquad \boldsymbol{t} = \begin{pmatrix} 36 \\ 12 \end{pmatrix}$$

[2 marks]

Question 17

Consider the vector,

$$\boldsymbol{R} = \begin{pmatrix} 13 \\ 16 \end{pmatrix}$$

Without using a calculator, show that $|\boldsymbol{R}| = 5\sqrt{17}$.

[1 mark]

Question 18
In general the 3 dimensional vector;

$$\boldsymbol{d} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

has a magnitude given by the formula;

$$|\boldsymbol{d}| = \sqrt{x^2 + y^2 + z^2}$$

Show that the magnitude of the following vector is an integer;

$$\boldsymbol{t} = \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix}$$

[1 mark]

Question 19
Try to find some more 3 dimensional vectors of integer components which have magnitudes which are also integers.

[1 marks]