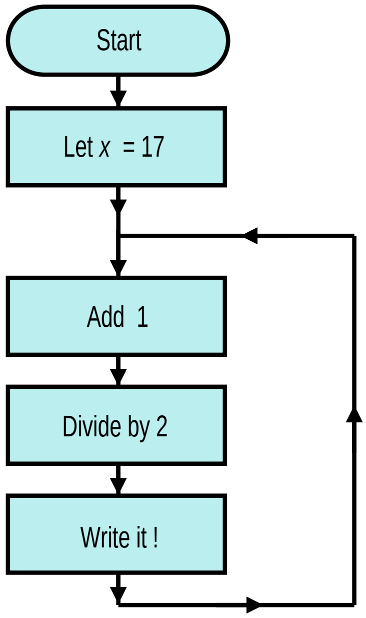
Lesson 2

Numerical Methods : Pure Year 2

2.1 A First Iteration

To iterate means to repeatedly carry out the same set of instructions.
Iteration is mathematical feedback.



In the flow chart, starting with 17 we repeatedly Add 1, Divide by 2 and Write it !

Thus:
9
5
3
2
1.5
1.25
1.125
1.0625
1.03125
1.015625
1.0078125
1.00390625
1.001953125

(i) This iteration is heading towards a fixed value.
What do you think the fixed value is ?

(ii) What happens if you start the iteration using the fixed value ?

This method of checking the fixed value only works for rational numbers.

2.2 Solving an equation using iteration.

Question :

$$g(x) = x^3 - 3x - 5$$

Use iteration to find, to 3 decimal places, a solution to $g(x) = 0$.

Solution : First we do some algebraic manipulation
(So algebra is not redundant, after all!)

$$\begin{aligned} g(x) &= 0 \\ x^3 - 3x - 5 &= 0 \\ x^3 &= 3x + 5 \\ x &= \sqrt[3]{3x + 5} \\ x_{n+1} &= \sqrt[3]{3x_n + 5} \end{aligned}$$

Let's start our iteration with $x_0 = 2$

$$\begin{aligned} x_0 &= 2 \\ x_1 &= 2.223980091 \\ x_2 &= 2.268372388 \\ x_3 &= 2.276967161 \\ x_4 &= 2.278623713 \\ x_5 &= 2.278942719 \\ x_6 &= 2.279004141 \end{aligned}$$

The iteration suggests that the solution to 3 decimal places is 2.279.

We confirm this by calculating that

$$\begin{aligned} g(2.2785) &= -0.0065254 \\ g(2.2795) &= 0.0060561 \end{aligned}$$

As $g(x)$ is NEGATIVE when $x = 2.2785$ and POSITIVE when $x = 2.2795$ there must be a value of x between 2.2785 and 2.2795 where the equation equals zero exactly.

As all values of x between 2.2785 and 2.2795 round off to to 2.279, the solution to 3 decimal places is 2.279.

2.3 Exercise

Question 1

The iteration formula for a sequence, C, is;

$$C_{n+1} = \frac{C_n}{2} + 3 \qquad with \qquad C_1 = 8$$

Calculate the first six terms of this sequence and then write down a guess of what you think the limit might be. Prove that your guess is correct by putting it into the iteration formula for C and observing what comes out.

Question 2

The iterative formula for a sequence, D, is;

$$D_{n+1} = \left(\sqrt{D_n} + 1\right)^2 \qquad with \qquad D_1 = 1$$

(i) Calculate D_2, D_3, D_4 and D_5.

(ii) This iteration is *divergent*.
Explain what *divergent* means.

(iii) You should recognise the sequence of numbers being generated.
What special name is given to these numbers ?

Question 3

The iteration formula for a sequence, E, is;

$$E_{n+1} = \frac{E_n}{4} + 6 \qquad with \qquad E_1 = 6$$

Calculate the first six terms of this sequence and then write down a guess of what you think the limit might be. Prove that your guess is correct by putting it into the iteration formula for E and observing what comes out.

Question 4

Here is an interesting iterative rule:

$$F_{n+1} = \frac{2}{F_n} - 1.$$

It is interesting because it has two limits.

(i) Show that - 2 is one limit and try to guess the other.

(ii) Let $F_1 = 2$.
Something goes horribly wrong !
What is it ?

Question 5

A sequence has the iteration formula;

$$G_{n+1} = \frac{1}{G_n} + 5 \qquad with \qquad G_1 = 2$$

(i) Calculate the values of G_2, G_3, G_4 ..... G_9 writing each to at least 8 decimal places.

(ii) What type of number is the limit this time ?

(iii) The perfect answer is actually

$$\frac{5 + \sqrt{a}}{2}$$

for some integer, a.

What is the value of a ?

Question 6

The *jumping tortoise* sequence has the iteration formula;

$$T_{n+1} = \frac{14}{T_n + 1.1} \qquad with \qquad G_1 = 10$$

Calculate sufficient values of T_2, T_3, T_4 ..... until you feel you know the fixed point of the iteration correct to five decimal places.

(i) Write down the fixed point correct to 5 decimal places.

(ii) Why do you think this is called the *jumping tortoise* sequence ?

Question 7

A famous sequence is described iteratively as follows,

$$F_1 = 1, \qquad F_2 = 1, \qquad F_{n+1} = F_n + F_{n-1}$$

(i) Write out the first ten terms of the sequence.

(ii) What is this sequence's famous name ?

Question 8

A sequence is described iteratively as follows,

$$S_1 = 1, \qquad S_2 = 1, \qquad S_{n+1} = S_n - S_{n-1}$$

(i) Write out the first ten terms of the sequence.

(ii) What is S_{33} ?

Question 9

The first five terms of the Fibonacci sequence are;

$F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$

(i) Write out the next five terms.

$F_6 =$, $F_7 =$, $F_8 =$, $F_9 =$, $F_{10} =$

[3 marks]

(ii) Work out each of the following;

(i) $F_1 \times F_3 - 1 =$

(ii) $F_2 \times F_4 + 1 =$

(iii) $F_3 \times F_5 - 1 =$

(iv) $F_4 \times F_6 + 1 =$

(v) $F_5 \times F_7 - 1 =$

(vi) $F_6 \times F_8 + 1 =$

(vii) $F_7 \times F_9 - 1 =$

(viii) $F_8 \times F_{10} + 1 =$

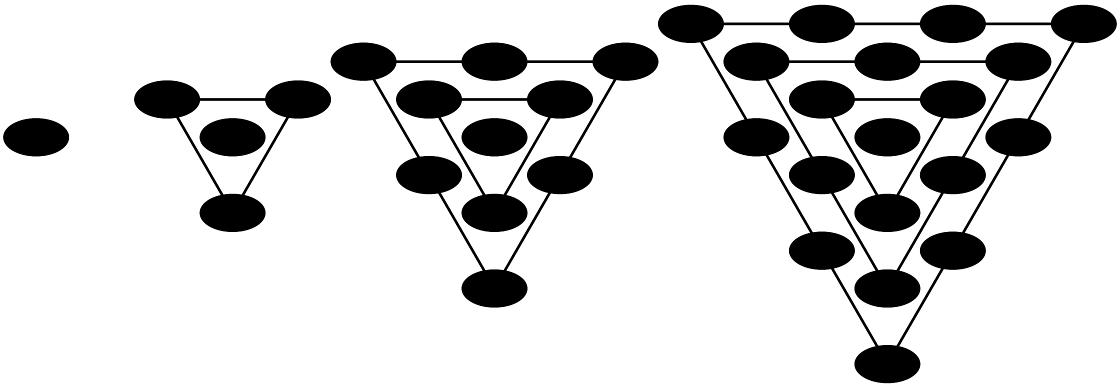
[3 marks]

(iii) Comment on the pattern.

[4 marks]

Question 10

Study the sequence of patterns below;



The sequence can be described iteratively;

$$G_1 = 1, \qquad G_2 = 4, \qquad G_{n+1} = 2G_n - G_{n-1} + 3$$

(i) Show how you would use the information $G_1 = 1$ and $G_2 = 4$ and the iterative relationship to calculate the value of G_3.

(ii) Show how you would use the iterative description to calculate G_4.

(iii) Write out the sequence starting with G_1 and up to G_{10}.