Lesson 7

Numerical Methods : Year 2

7.1 The Newton-Raphson Iteration Formula

Given an equation of the form

$$f(x) = 0$$

the Newton-Raphson iteration formula to find numerical solutions is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

In many cases this formula will result in swift convergence upon a root, especially if the starting value is chosen carefully.

However, in common with all iterative methods it can fail for a variety of reasons. For example,

- It may diverge rather than converge
- Convergence may be very slow
- The iteration may get stuck in a periodic loop
- A division by zero may crash the iteration
- The taking of a negative square root may crash the iteration
- The taking of a zero or negative logarithm may crash the iteration

Furthermore, because the Newton-Raphson formula works by 'firing tangents' at the x-axis it can fail if $f(x)$ has any points at which the derivative does not exist or if a tangent is 'fired' parallel to the x-axis, which makes turning points a source of iteration failure.

7.2 Example

(i) Sketch the graph of $y = \sin x$ and the graph of $y = 0.25x$ over $0 \leqslant x \leqslant 2\pi$

Consider the function

$$f(x) = \sin x - 0.25x$$

(ii) Write down the derivative of this function, $f'(x)$

A root of the equation $f(x) = 0$ is to be found by use of the Newton-Raphson iteration formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad x_0 = 3$$

(iii) Determine the values of the first 6 terms of the iterative sequence, writing your answers to 8 decimal places.

(iv) By considering the change of sign in $f(x)$ over an appropriate interval, show that a root is 2.475, correct to 3 decimal places.

7.3 Exercise

Question 1

(i) Sketch the graph of $y = e^x$ and the graph of $y = \frac{1}{x}$

Consider the function

$$f(x) = e^x - \frac{1}{x}$$

(ii) Write down the derivative of this function, $f\,'(x)$

A root of the equation $f(x) = 0$ is to be found by use of the Newton-Raphson iteration formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad x_0 = 1$$

(iii) Determine the values of the first 6 terms of the iterative sequence, writing your answers to 8 decimal places.

(iv) By considering the change of sign in $f(x)$ over an appropriate interval, show that a root is 0.567, correct to 3 decimal places.

Question 2

(i) Sketch the graph of $y = \sqrt{x}$ and the graph of $y = e^{-x}$

Consider the function

$$f(x) = \sqrt{x} - e^{-x}$$

(ii) Write down the derivative of this function, $f'(x)$

A root of the equation $f(x) = 0$ is to be found by use of the Newton-Raphson iteration formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad x_0 = 1$$

(iii) Determine the values of the first 6 terms of the iterative sequence, writing your answers to 8 decimal places.

(iv) By considering the change of sign in $f(x)$ over an appropriate interval, show that a root is 0.426, correct to 3 decimal places.

Question 3

(i) Sketch the graph of $y = x \sin x$ and the graph of $y = \cos x$, $0 \leqslant x \leqslant 2\pi$

Consider the function

$$f(x) = x \sin x - \cos x$$

(ii) Write down the derivative of this function, $f'(x)$

A root of the equation $f(x) = 0$ is to be found by use of the Newton-Raphson iteration formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad x_0 = 1$$

(iii) Determine the values of the first 6 terms of the iterative sequence, writing your answers to 8 decimal places.

(iv) By considering the change of sign in $f(x)$ over an appropriate interval, show that a root is 0.860, correct to 3 decimal places.

Question 4

(i) By writing $y = x^x$ in the form $\ln y = x \ln x$, show that

$$\frac{dy}{dx} = x^x (\ln x + 1)$$

[4 marks]

(ii) Show that the function $f(x) = x^x - 2$ has a root, β, in the interval [1.4, 1.6]

[2 marks]

(iii) Taking $x_0 = 1.5$ as a first approximation to β, apply the Newton-Raphson procedure once to obtain a second approximation to β, giving your answer to 4 decimal places

[4 marks]

(iv) By considering a change of sign of $f(x)$ over a suitable interval, show that $\beta = 1.5596$ correct to 4 decimal places

[3 marks]