

4.1 The Hole In The Sequence

Mathematicians like integer sequences, and they absolutely love it when there is a mysterious hole in a sequence; a term that should be there but isn't !

Here is a Calculus integer sequence of derivatives;

Function $f(x)$	Derivative $f'(x)$
...	...
$\frac{x^3}{3}$	x^2
$\frac{x^2}{2}$	x^1
$\frac{x^1}{1}$	x^0
The Hole	x^{-1}
$\frac{x^{-1}}{-1}$	x^{-2}
$\frac{x^{-2}}{-2}$	x^{-3}
...	...

The reason for the hole is that the function sequence is hitting a division by zero;

$$\dots, \frac{x^3}{3}, \frac{x^2}{2}, \frac{x^1}{1}, \frac{x^0}{0}, \frac{x^{-1}}{-1}, \frac{x^{-2}}{-2}, \dots$$

Fortunately, from our work on differentiation[†], it is known what is in the hole.

In other words, we know what function differentiates to $\frac{1}{x}$; it's $\ln(x)$.

Consequently, the statement of The Chain Rule Backwards can be extended to include this special case;

The Chain Rule Backwards

$$\int f'(x) [f(x)]^n dx = \frac{[f(x)]^{n+1}}{(n+1)} + c \quad n \neq -1$$

$$\int f'(x) [f(x)]^{-1} dx = \ln|f(x)| + c \quad \text{i.e. with } n = -1$$

[†] Differentiation III, Lesson 7

Example

Determine: $\int \frac{36x^2}{4x^3 - 9} dx$

Teaching Video: <http://www.NumberWonder.co.uk/v9045/4.mp4>



[3 marks]

4.2 Exercise

*Any solution based entirely on graphical
or numerical methods is not acceptable*

Marks Available : 30

Question 1

Determine: $\int \frac{40x^3}{1 + 2x^4} dx$

[3 marks]

Question 2

Determine: $\int \frac{35x^4}{3 - x^5} dx$

[3 marks]

Question 3

Determine: $\int \frac{x^3 + 1}{x^4 + 4x} dx$

[4 marks]

Question 4

Determine: $\int \frac{8x^3}{(x^2 + 1)(x^2 - 1)} dx$

[4 marks]

Question 5

- (i) Explain why finding $\int \frac{x + 3}{x^2 + x} dx$ can not be done by a straight forward application of The Chain Rule Backwards.

[2 marks]

- (ii) Prove that $\frac{x + 3}{x^2 + x} = \frac{3}{x} - \frac{2}{x + 1}$

Begin your proof “RHS =”

[2 marks]

- (iii) Use the part (ii) result to show $\int \frac{x + 3}{x^2 + x} dx = \ln \left| \frac{x^3}{(x + 1)^2} \right| + c$

[4 marks]

Question 6

Show that,

$$\int_1^3 \frac{6x^3 + 5x}{3x^4 + 5x^2 + 1} dx = \ln\left(\frac{17}{3}\right)$$

[8 marks]

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Teachers may obtain detailed worked solutions to the exercises by email from MHHShrewsbury@Gmail.com