**Lesson 3**

**A-Level Pure Mathematics : Year 2**
**Applications of Trigonometry**

## 3.1 Conversions Between Radians and Degrees

The key relationship between radians and degrees is the fact that

$$\pi^c = 180°$$

### 3.1.1 Radians $\rightarrow$ Degrees

If both sides of the key relationship are divided by $\pi$ we get that;

$$1^c = \left(\frac{180}{\pi}\right)^o \qquad \Rightarrow \qquad D = \frac{R \times 180}{\pi}$$

Thus, to convert radians $\rightarrow$ degrees, multiply by 180 and divide by $\pi$

### 3.1.2 Degrees $\rightarrow$ Radians

If both sides of the key relationship are divided by 180 we get that;

$$\left(\frac{\pi}{180}\right)^c = 1° \qquad \Rightarrow \qquad R = \frac{D \times \pi}{180}$$

Thus, to convert degrees $\rightarrow$ radians, multiply by $\pi$ and divide by 180

### 3.1.3 Example

Convert these small angles, given in radians, into degrees to one decimal place

| $\theta$ radians | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|---|---|---|---|---|---|
| $\theta$ degrees | | | | | |

**[ 5 marks ]**

## 3.2 An Investigation

Complete the following table, where $\theta$ is measured in radians.
Work to 3 decimal places

| $\theta^c$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|---|---|---|---|---|---|
| $\sin\theta$ | | | | | |
| $\cos\theta$ | | | | | |
| $\tan\theta$ | | | | | |
| $1 - \frac{\theta^2}{2}$ | | | | | |

**[ 5 marks ]**

### 3.3 Small Angle Approximations

The table suggests that when $\theta$ is small and measured in radians

- $\sin\theta \approx \theta$
- $\cos\theta \approx 1 - \dfrac{\theta^2}{2}$
- $\tan\theta \approx \theta$

By 'small' is meant either "relatively close to zero" or "much less that whatever else is around". The table suggests that when the angle has a size of more than half a radian (about 30 degrees) the approximation is becoming too inaccurate to be of practical use.

The angles at which the relative error exceeds 1% are;

$0.244^c$ or 14° for the *sin* $\theta$ approximation

$0.664^c$ or 38° for the *cos* $\theta$ approximation

and $0.176^c$ or 10° for the *tan* $\theta$ approximation

Although these small angle approximations are useful in some situations, they are not a substitute for trigonometric manipulations that provide exact solutions to problems regardless of the size of the angle.

### 3.4 Example

**( i )** When $\theta$ is small and measured in radians, show that

$$\frac{\sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta}{\theta} \approx 10$$

**( ii )** When $\theta = 0.1^c$ (about 6°) what is the percentage error introduced by using the small angle approximation for *sin* $\theta$ ?

Teaching Video : http://www.NumberWonder.co.uk/v9057/3.mp4

What the video.
Write out the solution.

☞

**[ 6 marks ]**

## 3.5 Exercise

*Any solution based entirely on graphical*
*or numerical methods is not acceptable*

Marks Available : 40

**Question 1**

**( i )** When $\theta$ is small and measured in radians, show that

$$\frac{4\sin 3\theta \ - \ \tan 2\theta}{2\theta} \approx 5$$

**[ 2 marks ]**

**( ii )** When $\theta = 0.1^c$ (about 6°) what is the percentage error introduced by using the small angle approximations ?

**[ 3 marks ]**

**Question 2**

**( i )** When $\theta$ is small and measured in radians, show that

$$\frac{\cos 4\theta \ - \ 1}{\theta \sin 2\theta} \approx -4$$

**[ 3 marks ]**

**( ii )** When $\theta = 0.1^c$ (about 6°) what is the percentage error introduced by using the small angle approximations ?

**[ 3 marks ]**

**Question 3**

*A-Level Paper 1 Examination Question from June 2018, Q1*

Given that $\theta$ is small and measured in radians, use the small angle approximations to find an approximate value of

$$\frac{1 - \cos 4\theta}{2\theta \sin 3\theta}$$

**[ 3 marks ]**

**Question 4**

**( i )** Show that, when $\theta$ is small and measured in radians,

$$\cos 2\theta - \sin^2 3\theta \approx 1 - 11\,\theta^2$$

**[ 2 marks ]**

**( ii )** Explain why, if $\theta$ is small in relation to the number 1, the term involving $\theta^2$ can be ignored.

**[ 1 mark ]**

**( iii )** Hence determine the percentage error in using the approximate value of $\cos 2\theta - \sin^2 3\theta$ when $\theta = 0.1^c$

**[ 2 marks ]**

**Question 5**

**( i )** Show that, when $\theta$ is small and measured in radians,

$$\frac{4\cos 3\theta - 2 + 5\sin\theta}{1 - \sin 2\theta}$$

can be written as

$$9\theta + 2$$

**[ 3 marks ]**

**( ii )** When $\theta = 0.2^c$ (about 11°) what is the percentage error introduced by using the small angle approximations ?

**[ 2 marks ]**

**Question 6**

**( i )** Use the small angle approximations to derive an expression for

$$\cos\theta + \tan^2\theta$$

when $\theta$ is small and measured in radians.

**[ 2 marks ]**

**( ii )** Hence explain why the small angle approximations give that

$$\cos\theta + \tan^2\theta \approx 1$$

when $\theta$ is small and measured in radians

**[ 1 mark ]**

**Question 7**

**( i )** Use the small angle approximations to solve the equation

$$5\,\sin 2\theta \;+\; \tan 3\theta \;=\; 0.39$$

**[ 2 marks ]**

**( ii )** In view of your part **( i )** answer, was the use of the small angle approximations valid ?
Explain your answer.

**[ 2 marks ]**

**Question 8**

**( i )** Show that, when $\theta$ is small and measured in radians,

$$\sin 5\theta \;-\; \cos 2\theta \;+\; \tan 2\theta$$

can be approximated by an equation of the form

$$a\,\theta^2 \;+\; b\,\theta \;+\; c$$

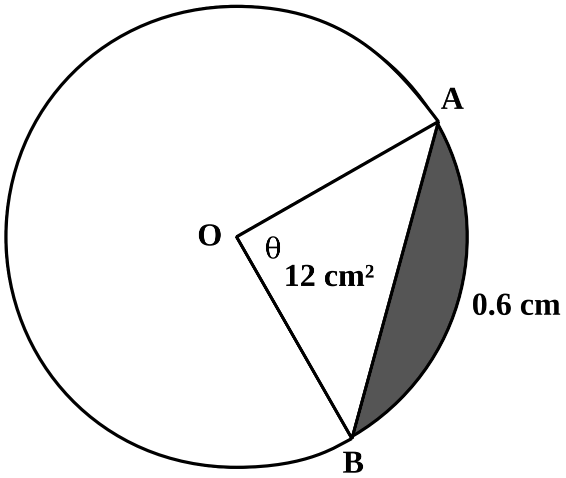
where $a$, $b$ and $c$ are integer constants to be found

**[ 3 marks ]**

**( ii )** Jason claims that the approximation will have a value of $c$ when $\theta$ is small because the terms in $\theta^2$ and $\theta$ can be ignored.
By NOT ignoring these terms show that Jason is wrong when $\theta = 0.1^c$

**[ 1 mark ]**

**Question 9**




**( i )** Determine the size of angle $\theta$, making use of the small angle approximations where appropriate.

**[ 4 marks ]**

**( ii )** Justify the use of the small angle approximations in deriving your part (i) answer

**[ 1 marks ]**