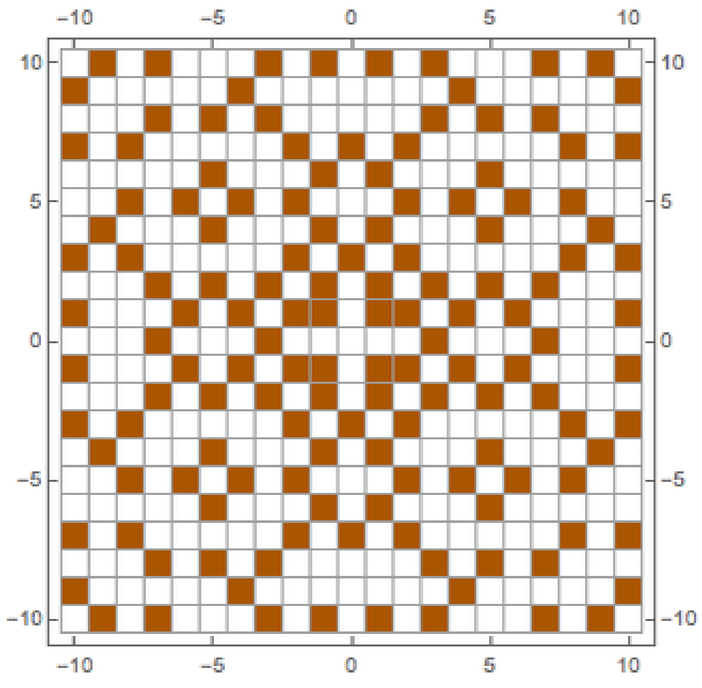
6.1 The Gaussian Integers

The Gaussian integers are complex numbers of the form $a + b\text{i}$ where both a and b are integers. For this subset of complex numbers unique factorisation holds. They are attractive to work with because each Gaussian integer is either a prime Gaussian integer or a composite Gaussian integer. A composite Gaussian integer can be decomposed into a unique product of prime Gaussian integers. So, some concepts, familiar from earlier work with the real integers, get a new lease of life with the Gaussian integers. The statement, $z \in \mathbb{Z}[\text{i}]$ says "z is a Gaussian integer".



The integer grid shows Gaussian primes for values around zero. On the real axis observe that 3 and 7 are Gaussian prime but 2 and 5 are not. What are the factors of these composite Gaussian integers 2 and 5 ? With a little thought,

- 2 factorises into $(1 + \text{i})(1 - \text{i})$
- 5 factorises into $(2 + \text{i})(2 - \text{i})$

From a careful look at the diagram, $(1 \pm \text{i})$ and $(2 \pm \text{i})$ are all Gaussian prime. There is no further factorising to do and the unique prime factorisation of 2 and 5 in Gaussian integers has been found.

In lesson 3, the number 6 was used to show that, in the general, complex numbers don't have unique factorisation because 6 could be factorized in more than one way,

- $6 = 3(1 + \text{i})(1 - \text{i})$
- $6 = \left(1 + \sqrt{5}\,\text{i}\right)\left(1 - \sqrt{5}\,\text{i}\right)$
- $6 = 2\left(\sqrt{2} + \text{i}\right)\left(\sqrt{2} - \text{i}\right)$

The second and third of these is not valid if only Gaussian integers are being considered and so this is an illustration that in $\mathbb{Z}[i]$ the composite number 6 does indeed have the unique factorisation,

- $6 = 3(1 + i)(1 - i)$

where each of these three factors is a Gaussian prime (easily checked on the grid).

6.2 The Square Root of a Complex Number

In the teaching video I will show you a very clever way to square root a Gaussian integer if it has an answer that is also a Gaussian integer.

Find z such that $z^2 - 5 - 12i = 0, \qquad z \in \mathbb{Z}[i]$

Teaching Video : http://www.NumberWonder.co.uk/Video/v9085(6).mp4

[6 marks]

I am indebted to Dr Neha Agrawal of Murray Edwards College, Cambridge University, for teaching me this method of finding complex square roots in $\mathbb{Z}[i]$

6.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 50

Question 1

A complex number $z = a + b\mathrm{i}$, where a and b are real numbers, is sought that satisfies the equation,

$$z^2 - 7 - 24\mathrm{i} = 0 \qquad z \in \mathbb{Z}[\mathrm{i}]$$

(i) Show that $ab = 12$

[2 marks]

(ii) Write down the value of $a^2 - b^2$

[1 mark]

(iii) Hence find the two values of z such that

$$z^2 - 7 - 24\mathrm{i} = 0 \qquad z \in \mathbb{Z}[\mathrm{i}]$$

[3 marks]

(iv) Show that your part (iii) solutions, when squared are equal to $7 + 24\mathrm{i}$ as you would expect.

[2 marks]

Question 2

A complex number $z = a + b\text{i}$, where a and b are real numbers, is sought that satisfies the equation,

$$z^2 + 9 - 40\text{i} = 0 \qquad z \in \mathbb{Z}[\text{i}]$$

(i) Show that $ab = 20$

[2 marks]

(ii) Write down the value of $a^2 - b^2$

[1 mark]

(iii) Hence find the two values of z such that

$$z^2 + 9 - 40\text{i} = 0 \qquad z \in \mathbb{Z}[\text{i}]$$

[3 marks]

(iv) Show that your part (iii) solutions, when squared, are equal to – 9 + 40i as you would expect.

[2 marks]

Question 3

A complex number $z = a + b\text{i}$, where a and b are real numbers, is sought that satisfies the equation,

$$z^2 - 40 + 42\text{i} = 0 \qquad z \in \mathbb{Z}[\text{i}]$$

(a) Find the two possible values of z

[5 marks]

(b) **(i)** The formula to solve quadratic equations of the form

$$a\,z^2 + bz + c = 0 \qquad z \in \mathbb{C}$$

where a, b and c are complex numbers is

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2\,a}$$

Use this formula to try and solve,

$$z^2 - 40 + 42\text{i} = 0 \qquad z \in \mathbb{Z}[\text{i}]$$

[2 marks]

(ii) Comment on the usefulness of this solution technique in this case.

[1 mark]

Question 4

(i) Work out $\left(1 + \sqrt{7}\,\mathrm{i}\right)\left(1 - \sqrt{7}\,\mathrm{i}\right)$

[1 mark]

(ii) Work out $(1 + \mathrm{i})(1 - \mathrm{i})(1 + \mathrm{i})(1 - \mathrm{i})(1 + \mathrm{i})(1 - \mathrm{i})$

[2 marks]

(iii) Explain how your part (i) and part (ii) answers show that complex numbers, in general, do not have unique factorisation.

[1 mark]

Question 5

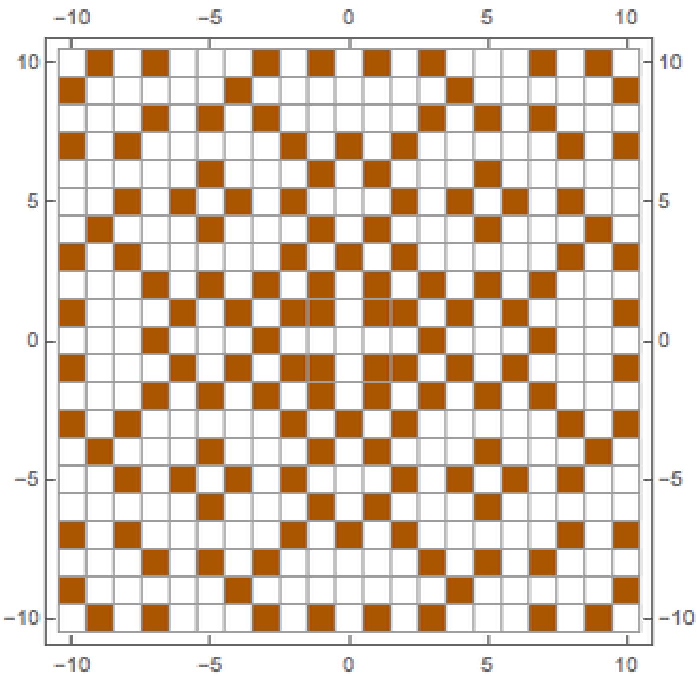
A certain number p is a prime integer in $\mathbb{R}$ but is not a Gaussian prime integer in $\mathbb{Z}[\mathrm{i}]$. When considered to be a Gaussian integer it is composite and has the unique factorization,

$$p = (3 + 2\mathrm{i})(3 - 2\mathrm{i})$$

What number is p ?

[1 mark]

Question 6



(i) On the grid of Gaussian integers, circle $6 + 5\mathrm{i}$

(ii) Hence state if $6 + 5\mathrm{i}$ is a Gaussian prime or not.

[1, 1 marks]

Question 7

A polynomial $p(z)$ has some coefficients that are complex numbers,

$$z^3 + \mathrm{i}\, z^2 + 6z = 0 \qquad z \in \mathbb{C}$$

The formula to solve quadratic equations of the form

$$a z^2 + bz + c = 0 \qquad z \in \mathbb{C}$$

where a, b and c are complex numbers is

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Show how this formula may be used to find the three roots of $p(z)$

[5 marks]

Question 8

A complex number $z = a + b\mathrm{i}$, where a and b are real numbers, is sought that satisfies the equation,

$$z^2 + 3 - 2\mathrm{i} = 0 \qquad z \in \mathbb{Z}[\mathrm{i}]$$

(i) Show that $ab = 1$

[2 marks]

(ii) Find the value of $a^2 - b^2$

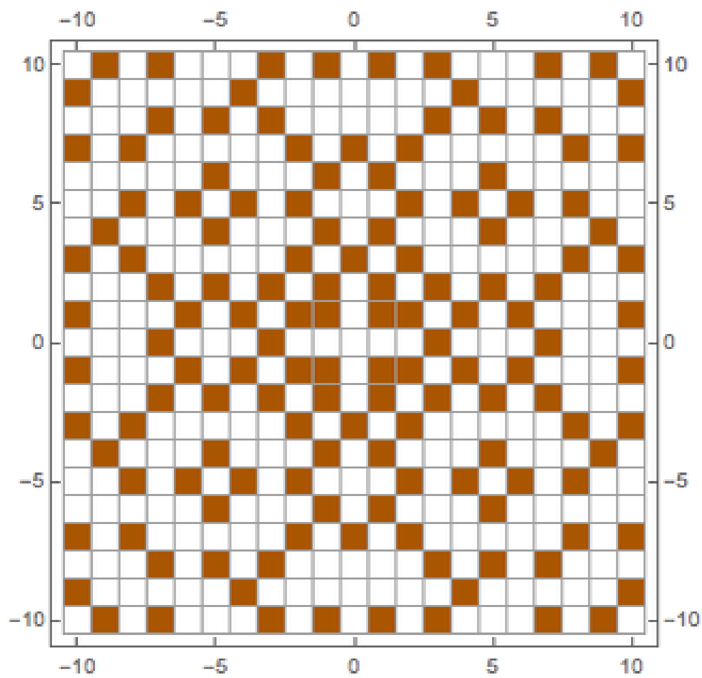
[1 mark]

(iii) Explain why there is no Gaussian integer that satisfies the equation

$$z^2 + 3 - 2\mathrm{i} = 0 \qquad z \in \mathbb{Z}[\mathrm{i}]$$

[2 marks]

(iv) On the grid of Gaussian integers, circle the complex number $-3 + 2\mathrm{i}$



[1 mark]

(v) Explain how your part (iv) answer confirms that there is no complex number that satisfies the equation $z^2 + 3 - 2\mathrm{i} = 0$, $z \in \mathbb{Z}[\mathrm{i}]$

[2 marks]

Question 9
A-Level Examination Question from June 2004, P4, Q3

The complex number $z = a + b\text{i}$, where a and b are real numbers, satisfies the equation,

$$z^2 + 16 - 30\text{i} = 0$$

(a) Show that $ab = 15$

[2 marks]

(b) Write down a second equation in a and b and hence find the roots of

$$z^2 + 16 - 30\text{i} = 0$$

[4 marks]