

Lesson 8

Further A-Level Pure Mathematics : Core 1 Complex Numbers I

8.1 When Circles Collide

Consider the following complex number equation with a view to plotting its locus on an Argand diagram,

$$|z + 2 - i| = |z - 4 + i|$$

From lesson 7 it is clear that,

$$|z + 2 - i| = \lambda$$

or, equivalently, $|z - (-2 + i)| = \lambda$

describes a circle with centre $(-2, 1)$ and radius λ on an Argand diagram.

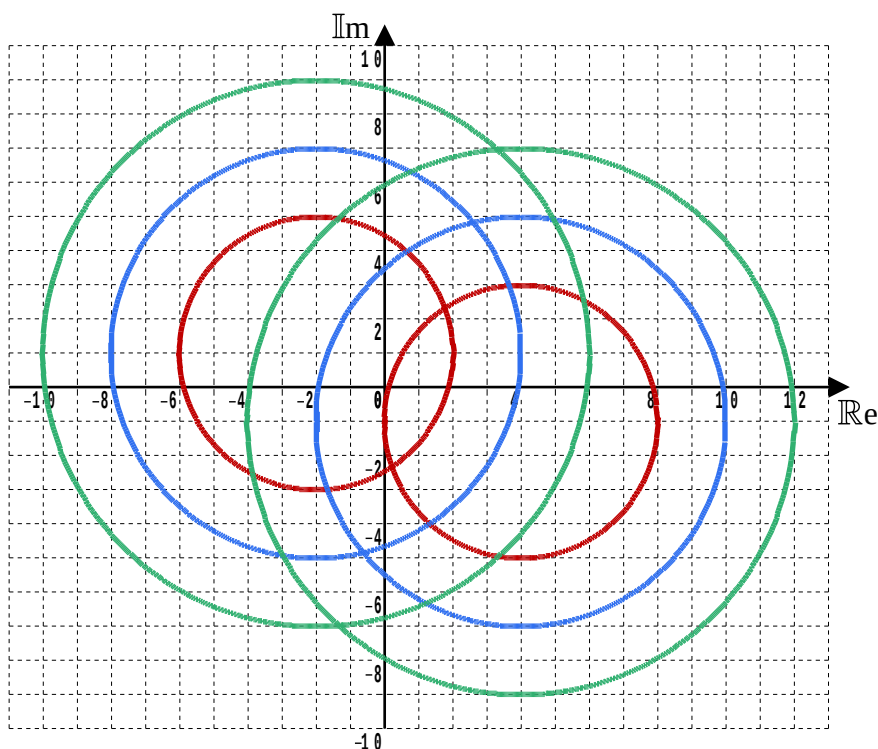
Similarly,

$$|z - 4 + i| = \lambda$$

or, equivalently, $|z - (4 - i)| = \lambda$

describes a circle with centre $(4, -1)$ and radius λ on an Argand diagram.

Equating the two can be thought of as generating a sequence of circles of the same radius, but with the radius increasing in steps, say $\lambda = 4, 6, 8, \dots$



The red circles are of radius 4, the blue of radius 6 and the green of radius 8.
The points of interest are where red meets red, blue meets blue and green meets green.

The Key Question

When those points of intersection are joined up what geometric shape is obtained ?

8.2 The Perpendicular Bisector

The quick answer to the key question is “a straight line”. However, more is true. From geometric work with a compass and straight edge, it can be seen that drawing a single pair of circles in this way is the method of obtaining a perpendicular bisector. The locus described by

$$|z + 2 - i| = |z - 4 + i|$$

is a line that cuts the line segment with endpoints at circle centres $(-2, 1)$ and $(4, -1)$ exactly in half and at right angles.

The Cartesian equation of this perpendicular bisector could be approximately read off from the intersecting circles diagram. Alternatively, as this very problem was tackled in the Year 1 course, we can use the mathematics already developed there.

Year 1 Revision Question

Find the Cartesian equation of the perpendicular bisector of line segment with endpoints $(-2, 1)$ and $(4, -1)$

Teaching Video : [http://www.NumberWonder.co.uk/Video/v9085\(8\).mp4](http://www.NumberWonder.co.uk/Video/v9085(8).mp4)



[3 marks]

8.3 An Alternative Method

The Cartesian equation of the perpendicular bisector described by,

$$|z + 2 - i| = |z - 4 + i|$$

can also be found by the following alternative method which uses the technique developed in lesson 7.

$$|z + 2 - i| = |z - 4 + i|$$

$$|x + yi + 2 - i| = |x + yi - 4 + i|$$

$$|(x + 2) + (y - 1)i| = |(x - 4) + (y + 1)i|$$

Take the modulus of each side, and square to remove the square roots

$$(x + 2)^2 + (y - 1)^2 = (x - 4)^2 + (y + 1)^2$$

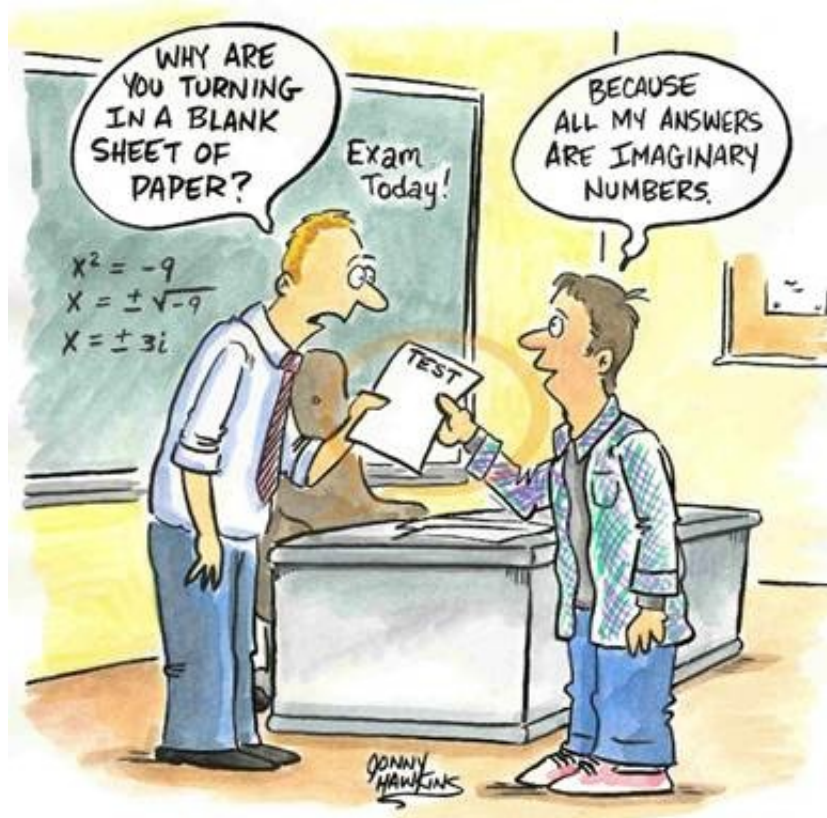
$$x^2 + 4x + 4 + y^2 - 2y + 1 = x^2 - 8x + 16 + y^2 + 2y + 1$$

$$4x - 2y + 5 = -8x + 2y + 17$$

$$12x - 12 = 4y$$

$$y = 3x - 3$$

This method will be required in the Exercise in Question 4 to get the locus of P and in Question 5 to get the requested equation of a circle.



8.4 Exercise

*Any solution based entirely on graphical
or numerical methods is not acceptable*

Marks Available : 50

Question 1

- (a) Write down the coordinates of the centre of the circle described by

$$|z - 10 - 5i| = \lambda \quad z \in \mathbb{C}, \lambda \in \mathbb{R}$$

[1 mark]

- (b) Write down the coordinates of the centre of the circle described by

$$|z + 6 - i| = \lambda \quad z \in \mathbb{C}, \lambda \in \mathbb{R}$$

[1 mark]

- (c) Hence, or otherwise, determine the Cartesian equation of the locus of z when

$$|z - 10 - 5i| = |z + 6 - i|$$

[3 marks]

- (d) What will be the shortest distance between the origin and the locus found in part (c) ? In other words, what is the minimum value of $|z|$?

A sketch of the locus on an Argand diagram may help !

[3 marks]

Question 2

Further A-Level Examination Question from June 2010, FP2, Q6 (edited)

A complex number z is represented by the point P in the Argand diagram.

(a) Given that,

$$|z - 6| = |z|$$

sketch the locus of P

[2 marks]

Another complex number w is represented by the point Q in the Argand diagram.

(b) Given that,

$$|w - 3 - 4i| = 5$$

add a sketch of the locus of Q to your part (a) sketch

[2 marks]

(c) Find the complex numbers which satisfy both

$$|z - 6| = |z| \quad \text{and} \quad |w - 3 - 4i| = 5$$

[2 marks]

Question 3

Sketch on the same Argand diagram the locus of points satisfying,

(a) $|z - 2i| = |z - 8i|$

[2 marks]

(b) $|z - 6 - 2i| = 5$

[3 marks]

The complex number z satisfies both,

$$|z - 2i| = |z - 8i|$$

and $|z - 6 - 2i| = 5$

(c) Use your answers to parts (a) and (b) to find the two values of z

[2 marks]

Question 4

Further A-Level Examination Question from June 2014, F2, Q7

The point P represents a complex number z on an Argand diagram, where

$$|z + 1| = |2z - 1|$$

and the point Q represents a complex number w on the Argand diagram, where

$$|w| = |w - 1 + i|$$

Find the exact coordinates of the points where the locus of P intersects the locus of Q

[7 marks]

Question 5

Further A-Level Exam Question from SAM, June 2017, FM Option 2, Paper 4, Q6

A curve has equation,

$$|z + 6| = 2|z - 6| \quad z \in \mathbb{C}$$

(a) Show that the curve is a circle with equation

$$x^2 + y^2 - 20x + 36 = 0$$

[2 marks]

(b) Sketch the curve on an Argand diagram

[2 marks]

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