

Lesson 2

Further A-Level Pure Mathematics : Core 1 Proof by Induction

2.1 Iterative Relations and Induction

A sequence is defined iteratively by the term-to-term formula,

$$u_1 = 1, \quad u_{n+1} = 3u_n + 4$$

The first few terms of this sequence are,

n	1	2	3	4	5	6	7	8
u_n	1	7	25	79	241	727	2185	6559

It is thought that a position-to-term formula is, $u_n = 3^n - 2$

It works for the eight terms listed, but to ensure it will always work a proof is needed.

Statement

For the sequence with the iterative definition, $u_1 = 1, \quad u_{n+1} = 3u_n + 4$

for all positive integers n , the n^{th} term is given by $u_n = 3^n - 2$

Proof by Induction

When $n = 1, \quad u_1 = 3^1 - 2$

$$= 1 \quad \text{which is the correct initial term}$$

Assume that when $n = k, \quad u_k = 3^k - 2$ in which case...

$$\begin{aligned} u_{k+1} &= 3u_k + 4 \\ &= 3(3^k - 2) + 4 \\ &= 3^{k+1} - 6 + 4 \\ &= 3^{k+1} - 2 \end{aligned}$$

Therefore, if the result is true for $n = k$, then it is true for $n = k + 1$

As the result has been shown to be true for $n = 1$, the conclusion is that it is true for all positive integers by mathematical induction $\square$

[5 marks]

The Teaching Video will talk you through the above proof.

Teaching Video : <http://www.NumberWonder.co.uk/v9096/2.mp4>



2.2 Exercise

*Any solution based entirely on graphical
or numerical methods is not acceptable*

Marks Available : 30

Question 1

A sequence is defined iteratively by the term-to-term formula,

$$u_1 = 1, \quad u_{n+1} = 2u_n + 1$$

The first few terms of this sequence are,

n	1	2	3	4	5	6	7	8
u_n	1	3	7	15	31	63	127	255

Prove by induction that the position-to-term formula is $u_n = 2^n - 1$

[5 marks]

Did you know ?

Numbers of the form $2^n - 1$ that are prime are called Mersenne Primes

Question 2

A sequence is defined iteratively by the term-to-term formula,

$$u_1 = 7, \quad u_{n+1} = 2u_n - 1$$

The first few terms of this sequence are,

n	1	2	3	4	5	6	7	8
u_n	7	13	25	49	97	193	385	769

Prove by induction that the position-to-term formula is $u_n = 3 \times 2^n + 1$

[5 marks]

Question 3

A sequence is defined iteratively by the hybrid formula,

$$u_1 = 6, \quad u_{n+1} = u_n + 2^n + 4 \quad n \geq 1$$

The first few terms of this sequence are,

n	1	2	3	4	5	6	7	8
u_n	6	12	20	32	52	88	156	288

Prove by induction that the position-to-term formula is $u_n = 2^n + 4n$

[5 marks]

Question 4

Further A-Level Examination Question from June 2013, IAL, FP1, Q9 (a)

A sequence of numbers is defined by

$$u_1 = 8$$

$$u_{n+1} = 4u_n - 9n \quad n \geq 1$$

Prove by induction that, for $n \in \mathbb{Z}^+$

$$u_n = 4^n + 3n + 1$$

[5 marks]

Question 5

Further A-Level Examination Question from January 2018, IAL, F1, Q8 (i) (Edexcel)

A sequence of numbers is defined by,

$$u_1 = 3$$

$$u_{n+1} = u_n + 3n - 2 \quad n \geq 1$$

Prove by induction, for all positive integers n

$$u_n = \frac{3}{2} n^2 - \frac{7}{2} n + 5$$

[5 marks]

Question 6

Further A-Level Examination Question, January 2013, IAL, FP1, Q8 (b) (Edexcel)

A sequence of positive integers is defined by,

$$u_1 = 1$$

$$u_{n+1} = u_n + n(3n + 1) \quad n \in \mathbb{Z}^+$$

Prove by induction that,

$$u_n = n^2(n - 1) + 1 \quad n \in \mathbb{Z}^+$$

[5 marks]

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Teachers may obtain detailed worked solutions to the exercises by email from mhh@shrewsbury.org.uk