

Further Pure A-Level Mathematics
Compulsory Course Component
Core 2

IMPROPER INTEGRALS



A three dimensional Möbius sculpture in solid aluminium
created by Sarang Sheth

IMPROPER INTEGRALS

Lesson 1

Further A-Level Pure Mathematics, Core 2

Improper Integrals

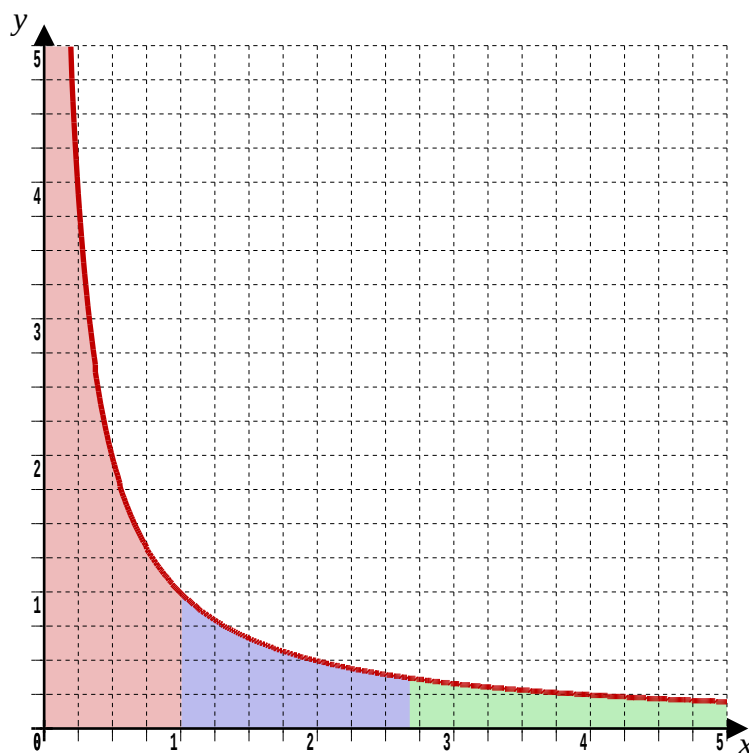
1.1 Divergent

Previously, our studies of integration have focussed on smooth, continuous and well behaved functions. It's time to tackle some of the issues that arise when trying to ease such restrictions and look at what are termed “Improper Integrals”.

As an example, consider the function graphed below, $y = \frac{1}{x}$, $x \neq 0$

We're going to consider the following three integrals,

$$(i) \quad \int_1^e \frac{1}{x} dx \quad (ii) \quad \int_0^1 \frac{1}{x} dx \quad (iii) \quad \int_e^\infty \frac{1}{x} dx$$



Before watching the teaching video, consider the following;

Part (i) of this example is a straightforward calculation of the area coloured blue.

Part (ii) has an issue with the vertical asymptote along the y-axis.

Is the area described by the integral (a piece is shown coloured pink) of finite value ?

Part (iii) has an issue with the horizontal asymptote along the x-axis.

Is the area described by the integral (a piece is shown coloured green) of finite value ?

Teaching Video: <http://www.NumberWonder.co.uk/v9100/1.mp4>



[6 marks]



A metal enamelled infinity brooch, of a snake eating itself

1.2 Exercise

*Any solution based entirely on graphical
or numerical methods is not acceptable*

Marks Available : 60

Question 1

- (i) Show that $\int_0^1 \frac{1}{x^2} dx$ is divergent.

Make sure you show the limiting process clearly in your working.

[4 marks]

- (ii) Show that $\int_1^\infty \frac{1}{x^2} dx$ is convergent and find its value.

Make sure you show the limiting process clearly in your working.



[4 marks]

Question 2

- (i) Show that $\int_0^1 \frac{1}{\sqrt{x}} dx$ is convergent and find its value.

Make sure you show the limiting process clearly in your working.

[4 marks]

- (ii) Show that $\int_1^\infty \frac{1}{\sqrt{x}} dx$ is divergent.

Make sure you show the limiting process clearly in your working.

[4 marks]

Question 3

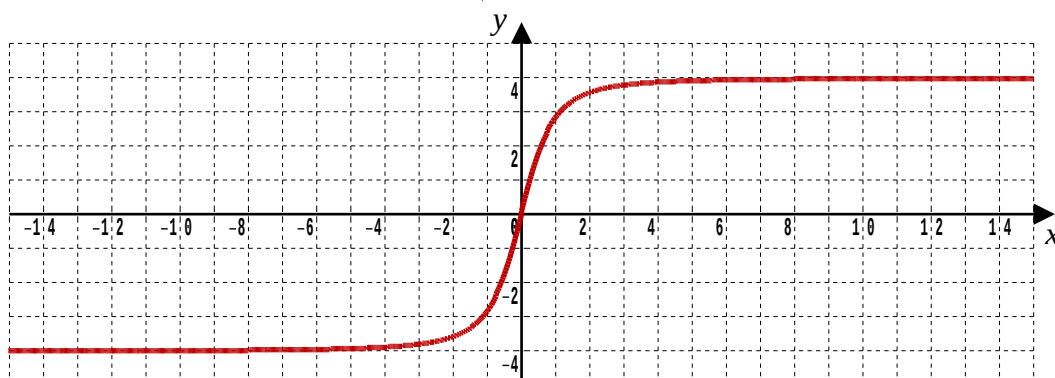
- (i) Given that $y = \sqrt{1 + x^2}$ find $\frac{dy}{dx}$

[2 marks]

- (ii) Hence, or otherwise, show that $\int_0^{\infty} \frac{4x}{\sqrt{1 + x^2}} dx$ is divergent.
Make sure you show the limiting process clearly in your working.

[4 marks]

- (iii) The graph of $f(x) = \frac{4x}{\sqrt{1 + x^2}}$ is shown below.



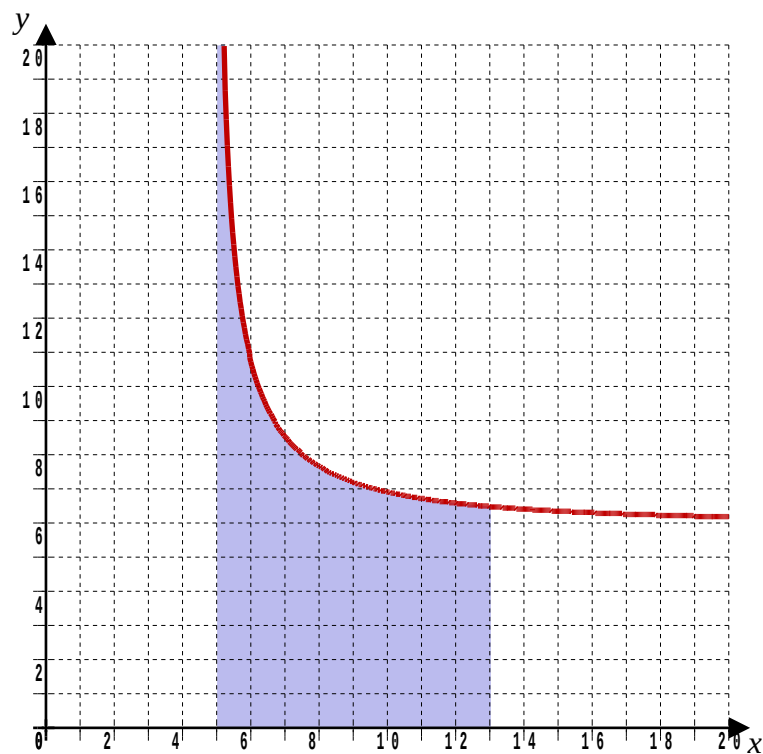
What features of the graph suggest that the part (ii) integral would be divergent ?

[2 marks]

Question 4

The graph is of the function,

$$f(x) = \frac{6x}{\sqrt{x^2 - 25}} \quad x \neq \pm 5$$



Show that $\int_5^{13} \frac{6x}{\sqrt{x^2 - 25}} dx$ converges and find its value.

[6 marks]

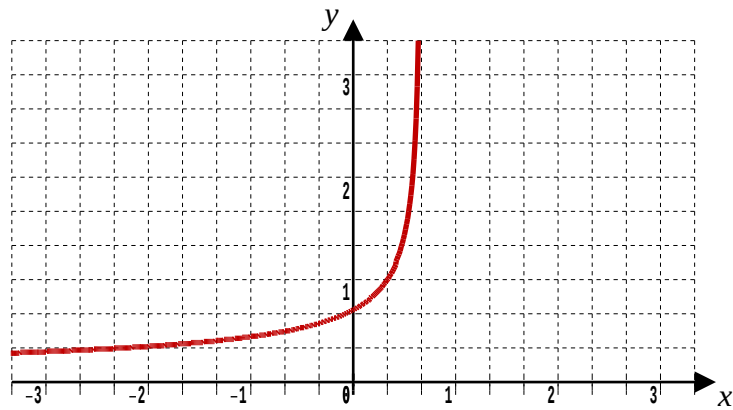
Question 5

The graph is of the function $f(x) = \frac{1}{\sqrt{2-3x}}$ $x \neq \frac{2}{3}$

Integration is to be used to find the area between this curve, the x -axis, the y -axis

and the vertical asymptote $x = \frac{2}{3}$

(i) On the graph shade in the area to be found.



[1 mark]

(ii) Explain why the integral involved is an “improper integral”.

[1 mark]

(iii) Show that $\int_0^{\frac{2}{3}} \frac{1}{\sqrt{2-3x}} dx$ converges and find its value.

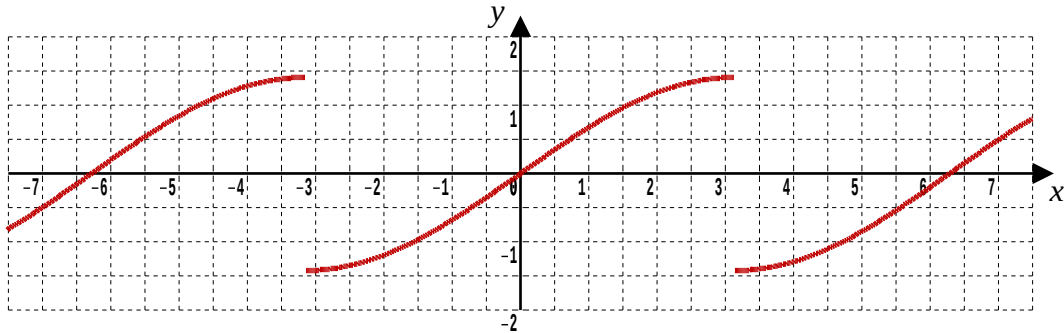
[6 marks]

Question 6

The graph is of the function $f(x) = \frac{\sin x}{\sqrt{1 + \cos x}}$ $x \neq (2n + 1)\pi$

Integration is to be used to find the area between this curve, the x -axis, the y -axis and the line $x = \pi$

(i) On the graph shade in the area to be found.



[1 mark]

(ii) Explain why the integral involved is an “improper integral”.

[1 mark]

(iii) Show that $\int_0^{\pi} \frac{\sin x}{\sqrt{1 + \cos x}} dx$ converges and find its value.

[6 marks]

Question 7

The graph is of the function $f(x) = \frac{3(x^2 + 1)}{\sqrt{4 - 3x - x^3}}$ $x \neq 1$

Integration is to be used to find the area between this curve, the x -axis, the y -axis and the vertical asymptote at $x = 1$

(i) On the graph shade in the area to be found.



[1 mark]

(ii) Explain why the integral involved is an “improper integral”.

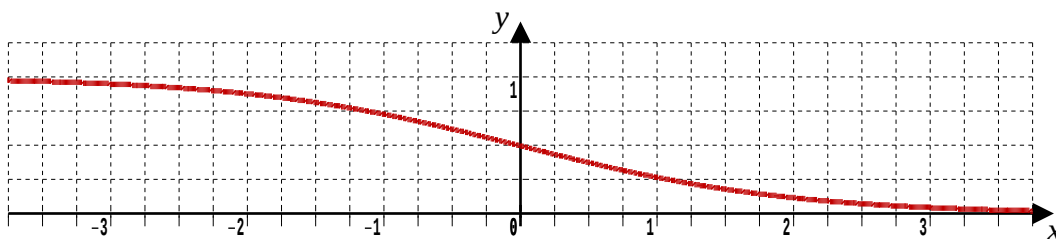
[1 mark]

(iii) Show that $\int_0^1 \frac{3(x^2 + 1)}{\sqrt{4 - 3x - x^3}} dx$ converges and find its value.

[6 marks]

Question 8

The graph is of the function $f(x) = 1 - \frac{1}{e^{-x} + 1}$



If the following integral is convergent find its value, otherwise prove it's divergent,

$$\int_0^{\infty} \left(1 - \frac{1}{e^{-x} + 1} \right) dx$$

[6 marks]