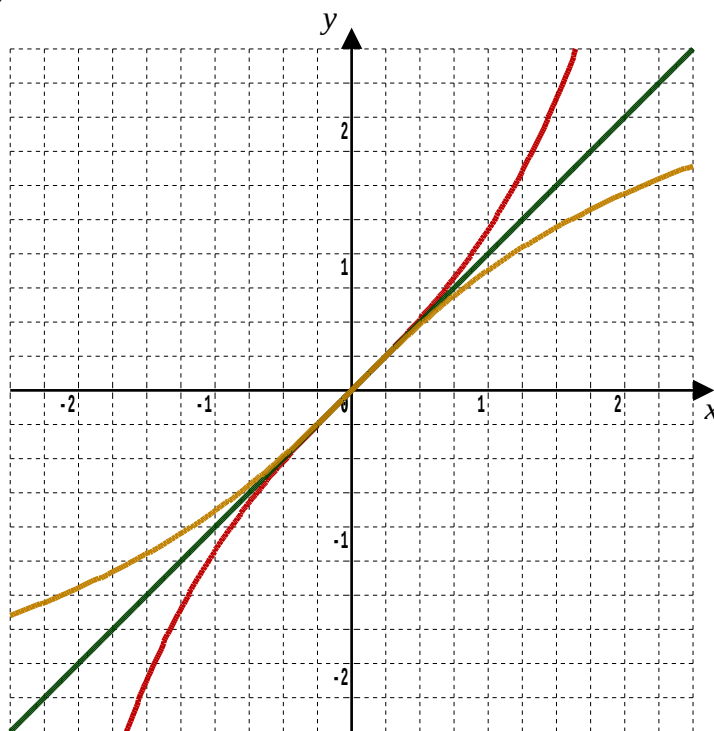


Lesson 3

Further A-Level Pure Mathematics, Core 2 Hyperbolic Functions

3.1 $\operatorname{arsinh} x$

Any one-one function, such as $\sinh x$, has an inverse that, graphically, is a reflection in the line $y = x$. The inverse of $\sinh x$ is called $\operatorname{arsinh} x$.



Red : $y = \sinh x$ Green : $y = x$ Gold : $y = \operatorname{arsinh} x$

Given that $\sinh x$ is defined in terms of exponentials, the expectation would be that the inverse function, $\operatorname{arsinh} x$, involves logarithms and this is the case.

The Inverse Of $\sinh x$: $\operatorname{arsinh} x$

$$\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2 + 1}\right) \quad x \in \mathbb{R}$$

A proof of this is written out on the next page which the following eight minute excellent video from *Exam Solutions* will talk through.

Teaching Video: <http://www.NumberWonder.co.uk/v9102/3.mp4>



3.2 The Proof

$$y = \operatorname{arsinh} x$$

$$\therefore x = \sinh y$$

$$= \frac{e^y - e^{-y}}{2} \quad \text{From the definition of } \sinh$$

$$2x = e^y - e^{-y}$$

$$2x e^y = (e^y)^2 - 1 \quad \text{From multiplying through by } e^y$$

$$(e^y)^2 - 2x(e^y) - 1 = 0 \quad \text{Which is a "quadratic in disguise"}$$

$$e^y = \frac{2x \pm \sqrt{(-2x)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{2x \pm \sqrt{4x^2 + 4}}{2}$$

$$= \frac{2x \pm \sqrt{4} \sqrt{x^2 + 1}}{2}$$

$$= x \pm \sqrt{x^2 + 1}$$

$$\text{Now, } e^y > 0 \text{ since } \sqrt{x^2 + 1} > x$$

$$\therefore e^y = x + \sqrt{x^2 + 1}$$

$$y = \ln(x + \sqrt{x^2 + 1})$$

$$\text{That is, } \operatorname{arsinh} x = \ln(x + \sqrt{x^2 + 1}) \quad \square$$

3.3 Example

Determine the exact value of

(i) $\operatorname{arsinh}(2)$ (ii) $\operatorname{arsinh}(-3)$

[2 marks]

Solution :

(i) $\operatorname{arsinh}(2) = \ln(2 + \sqrt{5})$ About 1.44

(ii) $\operatorname{arsinh}(-3) = \ln(-3 + \sqrt{10})$ About -1.81

This example is also covered in the teaching video.

3.4 Exercise

*Any solution based entirely on graphical
or numerical methods is not acceptable*

Marks Available : 30

Question 1

Express each of the following as a natural logarithm,

(i) $\operatorname{arsinh}(1)$

[1 mark]

(ii) $\operatorname{arsinh}(\sqrt{3})$

[1 mark]

(iii) $\operatorname{arsinh}(2\sqrt{2})$

[2 marks]

Question 2

Find the exact value of $\operatorname{arsinh}\left(\frac{3}{4}\right)$ in as simple a form as possible

[2 marks]

Question 3

Find the exact value of $\operatorname{arsinh}\left(\frac{1}{\sqrt{3}}\right)$ in as simple a form as possible

[2 marks]

Question 4

With the aid of the identity $\cosh^2 x - \sinh^2 x = 1$ solve the following equation, giving exact answers as natural logarithms.

$$2 \cosh^2 x - 5 \sinh x = 5$$

[6 marks]

Question 5

Further A-Level Examination Question from June 2015, FP3, Q1 (Edexcel)

Solve the equation

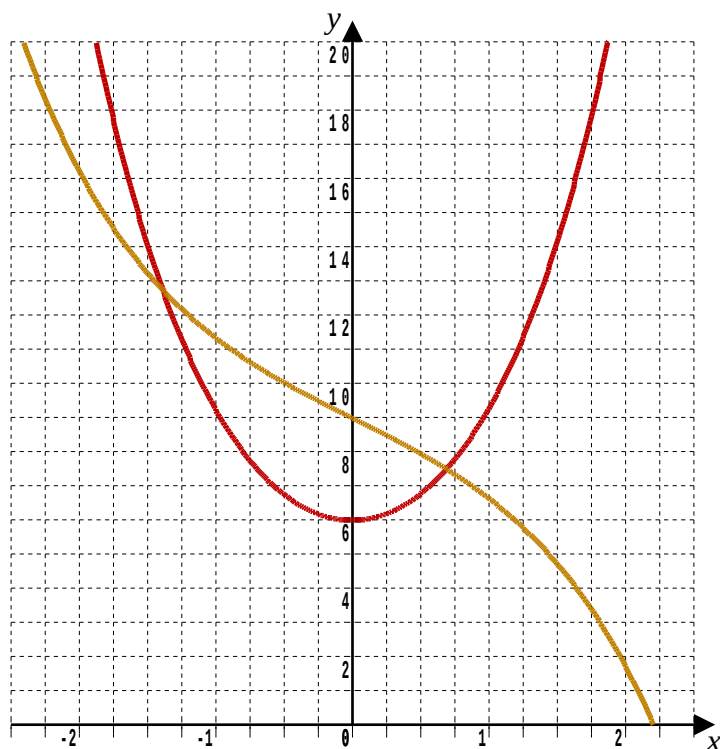
$$2 \cosh^2 x - 3 \sinh x = 1$$

giving your answers in terms of natural logarithms

[6 marks]

Question 6

Further A-Level Examination Question from June 2013, FP3(R), Q7(a) (Edexcel)



The curves have equations, $y = 6 \cosh x$ and $y = 9 - 2 \sinh x$

Using the definitions of $\sinh x$ and $\cosh x$ in terms of e^x , find exact values for the x -coordinates of the two points where the curves intersect.

[6 marks]

Question 7

Further A-Level Examination Question from June 2002, P6, Q1 (Edexcel)

Prove that $\sinh(i\pi - \theta) = \sinh \theta$

[4 marks]

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Teachers may obtain detailed worked solutions to the exercises by email from MHHShrewsbury@Gmail.com