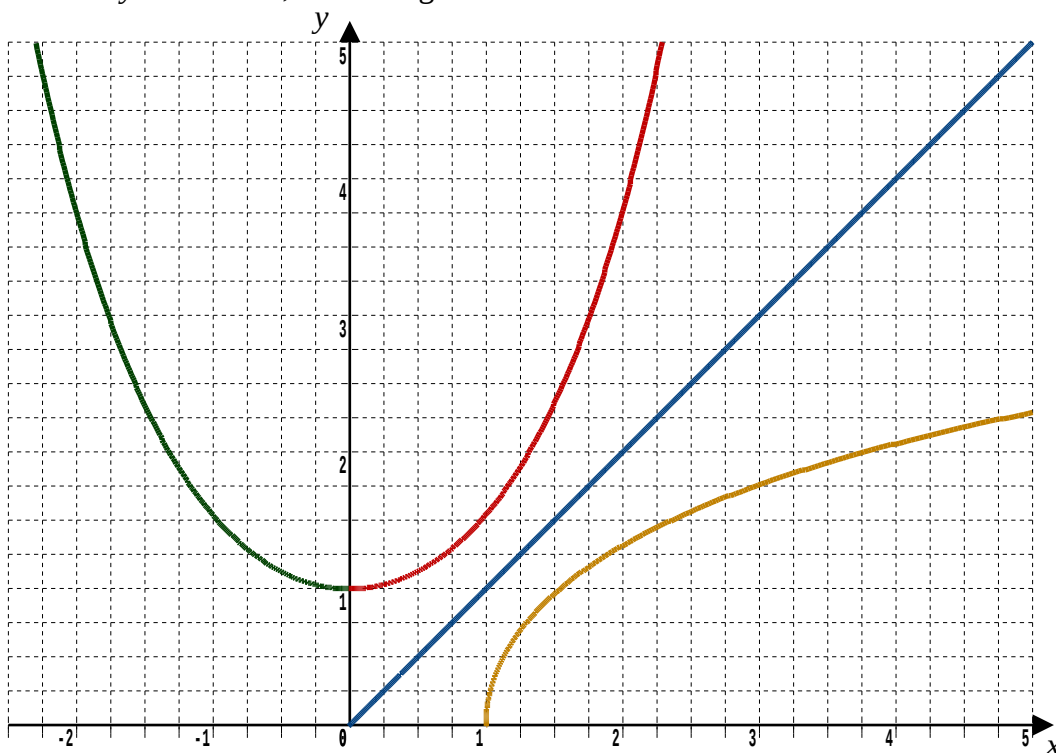


Lesson 5

Further A-Level Pure Mathematics, Core 2 Hyperbolic Functions

5.1 $\operatorname{arcosh} x$

Before the inverse of the $\cosh$ function can be found, the domain must be restricted to obtain a one-to-one function. On the graph below the curve $y = \cosh x$ is coloured green and red, the green part being excluded by the restriction on the domain $x \geq 0$. The red part is then reflected in the line $y = x$ to obtain $y = \operatorname{arcosh} x$, coloured gold.



The Inverse Of $\cosh x$: $\operatorname{arcosh} x$

$$\operatorname{arcosh} x = \ln \left(x + \sqrt{x^2 - 1} \right) \quad x \in \mathbb{R}, x \geq 1$$

A proof is on the next page and talked through in the teaching video from *Exam Solutions*. It's similar to the arsinh proof but with a few extra technicalities.

Teaching Video: <http://www.NumberWonder.co.uk/v9102/5.mp4>



5.2 The Proof

$$y = \operatorname{arcosh} x$$

$$\therefore x = \cosh y$$

$$= \frac{e^y + e^{-y}}{2} \quad \text{From the definition of } \cosh$$

$$2x = e^y + e^{-y}$$

$$2x e^y = (e^y)^2 + 1 \quad \text{From multiplying through by } e^y$$

$$(e^y)^2 - 2x(e^y) + 1 = 0 \quad \text{Which is a "quadratic in disguise"}$$

$$e^y = \frac{2x \pm \sqrt{(-2x)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{2x \pm \sqrt{4x^2 - 4}}{2}$$

$$= \frac{2x \pm \sqrt{4} \sqrt{x^2 - 1}}{2}$$

$$= x \pm \sqrt{x^2 - 1}$$

$$y = \ln(x \pm \sqrt{x^2 - 1})$$

Now, as the *arcosh* function is one-to-one only one of these two possibilities can apply. The one to select, corresponding to the gold curve that in turn came

from the red half of the *cosh* curve is the following,

$$\operatorname{arcosh} x = \ln(x + \sqrt{x^2 - 1}) \quad x \in \mathbb{R}, x \geq 1 \quad \square$$

(The other solution would correspond to the inverse of the green half of the *cosh* curve, and would be a reflection of the gold curve in the x-axis)

5.3 Example

- (i) Determine the exact value of $\operatorname{arcosh}(4)$
- (ii) Solve, to three decimal places, $\cosh w = 2$

[3 marks]

Solution :

$$(i) \quad \operatorname{arcosh}(4) = \ln(4 + \sqrt{15}) \quad \text{About } 2.06$$

$$\begin{aligned} (ii) \quad w &= \operatorname{arcosh}(2) \quad \text{i.e. } x = 2 \\ &= \ln(x \pm \sqrt{x^2 - 1}) \\ &= \ln(2 \pm \sqrt{3}) \\ &= \pm 1.316 \end{aligned}$$

This example is also covered in the *Exam Solutions* teaching video.

5.4 Exercise

*Any solution based entirely on graphical
or numerical methods is not acceptable*

Marks Available : 30

Question 1

Express each of the following as a natural logarithm,

(i) $\operatorname{arcosh}(3)$

[1 mark]

(ii) $\operatorname{arcosh}(\sqrt{3})$

[1 mark]

(iii) $\operatorname{arcosh}(3\sqrt{2})$

[2 marks]

Question 2

Find the exact value of $\operatorname{arcosh}\left(\frac{13}{5}\right)$ in as simple a form as possible

[3 marks]

Question 3

Solve the equation $\cosh w = 12$ giving both solutions to 3 decimal places

[3 marks]

Question 4

Further A-Level Examination Question from June 2017, FP3, Q3 (Edexcel)

(a) Using the definitions for $\cosh x$ in terms of exponentials, show that

$$\cosh 2x = 2 \cosh^2 x - 1$$

[3 marks]

(b) Find the exact values of x for which

$$29 \cosh x - 3 \cosh 2x = 38$$

giving your answers in terms of natural logarithms

[6 marks]

Question 5

Further A-Level Examination Question from June 2008, FP2, Q2 (Edexcel)

Find the values of x for which

$$8 \cosh x - 4 \sinh x = 13$$

giving your answers as natural logarithms.

[6 marks]

Question 6

Further A-Level Examination Question from June 2004, P5. Q1(b) (Edexcel)

Solve,

$$\operatorname{csch} x - 2 \operatorname{coth} x = 2$$

giving your answer in the form $k \ln a$ where k and a are integers

[5 marks]

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Teachers may obtain detailed worked solutions to the exercises by email from MHHShrewsbury@Gmail.com