

Lesson 2

A-Level Pure Mathematics, Year 2 Proof I : The Art of Absolute Certainty

2.1 Divisibility

In mathematics, seemingly simple results can have far reaching consequences. One such result is the following.

The Sum Divisor Rule

Let x , y and n be three natural numbers.

If both $(x + y)$ and x are divisible by n then y must also be divisible by n .

Proof

Let the sum by x and y be z where z will be another natural number.

$$x + y = z$$

Both x and z are divisible by n so I can write that $x = na$ and $z = nb$ for some natural numbers a and b , giving,

$$na + y = nb$$

$$y = nb - na$$

$$= n(b - a)$$

which shows that y is divisible by n as the rule claims $\square$

2.2 Divisibility by 2

It is easy to tell if a large number is even or odd simply by looking at the rightmost digit, R , of the number.

If $R \in \{0, 2, 4, 6, 8\}$ then the large number is even.

If $R \in \{1, 3, 5, 7, 9\}$ then the large number is odd.

But why does this test work ?

Observe that any natural number, $\mathbb{N}$, can be expressed in algebra as aR where R is the single rightmost digit, and a is all the other digits.

Then,
$$aR = 10a + R$$

By the Sum Divisor Rule, when aR and $10a$ are both divisible by 2, so must R be.

In other words when $R \in \{0, 2, 4, 6 \text{ or } 8\}$, aR is even, otherwise odd.



Divisibility : How to count the number of sheep in a field.

Step 1 : Count the number of legs.

Step 2 : Divide by four.

2.3 Another Divisibility Test

The test for whether or not a number is divisible by 2 begs the question, “Are there similar, straightforward techniques to determine if a large number is divisible by other primes” ?

The answer is that there are, although they are not as well known.

Divisibility by 7

Subtract twice the last digit from the number formed by the remaining digits.

Repeat as necessary.

If, at any stage, a result is obtained that is obviously divisible by 7 then the original number is also divisible by 7

2.3.1 Example #1

Without using a calculator, show that 1603 is divisible by 7

[2 marks]

$$1603 \Rightarrow 160 - 6 = 154$$

$$\Rightarrow 15 - 8 = 7 \text{ which is obviously divisible by 7}$$

$$\therefore 1603 \text{ is divisible by 7}$$

[2 marks]

2.3.2 Example #2

Without using a calculator, show that 504091 is divisible by 7

[2 marks]

2.3.3 Example #3

Provide a proof for the test for divisibility by 7

[7 marks]

2.4 Another Divisibility Rule

In the proof of the divisibility by 7 test, use was made of the Product Divisor Rule, which you may feel is obvious.

The Product Divisor Rule

Let x , y and n be three natural numbers.

If xy is divisible by n and x is not divisible by n then y must be divisible by n .

Proof

Let the product of x and y be z where z will be another natural number.

$$xy = z$$

As z is divisible by n , I can write that $z = na$ for some $a \in \mathbb{N}$.

$$xy = na$$

$$\frac{xy}{n} = a$$

As a cannot be a fraction, one of x or y must be writable as nb for some $b \in \mathbb{N}$ so that the n on the denominator can be cancelled with an n on the numerator.

But x is not divisible by n and so $x \neq nb$.

Therefore, $y = nb$ which shows y must be divisible by n as claimed. □

2.5 Exercise

*Any solution based entirely on graphical
or numerical methods is not acceptable*

Marks Available : 40 marks

Question 1

(i) Show that 24094 is divisible by 7 using the divisibility by 7 test.

[2 marks]

(ii) Explain why, after part (a), you immediately know that 24094 is divisible by 14

[2 marks]

Question 2

Young schoolchildren often think, mistakenly, that 91 is a prime number.
Show that 91 is divisible by 7 using the divisibility by 7 test.

[1 mark]

Question 3

The four digit numbers 5775, 3223 and 6996 are all of the form $abba$ where a and b are natural numbers between 0 and 9 inclusive.

(That is $a, b \in \mathbb{N}$, $0 \leq a \leq 9$, $0 \leq b \leq 9$)

Prove that all such four digit numbers of the form $abba$ are divisible by 11.

Hint : $abba = 1000a + 100b + 10b + a$

[4 marks]

Question 4

Prove that the three digit number aba when added to the three digit number bab is always divisible by 37

[5 marks]

Question 5

- (i) Prove that for any four digit number $abcd$ to be divisible by 3, the sum of its digits must be divisible by 3

[5 marks]

- (ii) Hence explain how you know that 7458 is divisible by 3

[2 marks]

Question 6

- (i) Prove that, for any natural number of more than two digits to be divisible by 4, the two rightmost digits must be divisible by 4

[6 marks]

- (ii) Hence explain how you know that 290547452 is divisible by 4

[2 marks]

Question 7

Divisibility by 13

Add four times the last digit to the number formed by the remaining digits.

Repeat as necessary.

If at any stage a result is obtained that is obviously divisible by 13 then the original number is divisible by 13

- (i) Without using a calculator, show that 119483 is divisible by 13

[2 marks]

- (ii) Without using a calculator, show that 504088 is divisible by 13

[2 marks]

- (iii) Provide a proof for the test for divisibility by 13

[7 marks]

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