5.1 Proof by Contradiction

Proof by contradiction is a clever method of proving that a statement is true by assuming that if the opposite were true, this would leads to a nonsensical situation. Given that this opposite is not true, the original statement must be true.

This piece of logical thinking takes a little bit of getting used to, and so we will look at three classic examples.

5.1.1 Example #1

Use proof by contradiction to show that there exist no integers a and b for which

$$21a + 14b = 1$$

Proof

Assume the opposite is true, integers a and b exist for which $21a + 14b = 1$

Divide both sides by the highest common factor of 21 and 14, that is, 7 to get,

$$3a + 2b = \frac{1}{7}$$

But this clearly cannot be, as any combination of integers that are multiplied, added and subtracted can only yield an integer answer, not a fraction.

Therefore, the original assumption, that integers a and b exist, must be false.

And so, there exist no integers a and b for which $21a + 14b = 1$ □

[3 marks]

5.1.2 Example #2

Use proof by contradiction to show that if n^2 is even, then n must be even.

Proof

[3 marks]

5.1.3 Example #3

Use proof by contradiction to show that $\sqrt{2}$ is an irrational number.

Proof

Firstly, to be clear, a number n is rational if it can be written in the form $\dfrac{p}{q}$ where p and q are integers, and q is not equal to zero. Mathematicians' write, $n \in \mathbb{Q}$.

An irrational number is a number that cannot be written in this form.

Assume the opposite of what is wanted is true; that $\sqrt{2}$ is a rational number. Then, by the definition of what a rational number is,

$$\sqrt{2} = \frac{p}{q} \text{ for some } p,\ q \in \mathbb{Z},\ q \neq 0$$

Furthermore, assume that this fraction cannot be reduced further, p and q have no factors in common. In other words, they are coprime with $hcf\ \{p,\ q\} = 1$

In consequence, $\quad 2 = \dfrac{p^2}{q^2}$

$$p^2 = 2\,q^2$$

The 2 on the RHS tells us that it is even. Therefore the LHS must be even.

From example #2, if p^2 is even then p is also even and so can be expressed in the form $p = 2a$ for some integer a. So we now have that,

$$(2a)^2 = 2\,q^2$$

$$4\,a^2 = 2\,q^2$$

$$2\,a^2 = q^2$$

The 2 on the LHS tells us that it is even. Therefore the RHS must be even.

Again, from example #2, if q^2 is even, then q is also even.

If p and q are both even, they will have a common factor of 2,

This contradicts the statement that p and q have no common factors.

The inescapable conclusion is that $\sqrt{2}$ is an irrational number. $\square$

[6 marks]

The Inescapable Conclusion

5.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40 marks

Question 1

For $x, y \in \mathbb{N}$, $x \neq y$, use proof by contradiction to show that,

$$\frac{x}{y} + \frac{y}{x} > 2$$

Hint : Assume that the opposite is true, that $\frac{x}{y} + \frac{y}{x} \leqslant 2$

Do algebra on this inequality to derive a statement that can't be true.

[3 marks]

Question 2

For $x \in \mathbb{R}$, $0 < x < 1$, use proof by contradiction to show that,

$$\frac{1}{x(1 - x)} \geqslant 4$$

Hint : From $0 < x < 1$ it follows that $x > 0$ and $(1 - x) > 0$ and so $x(1 - x) > 0$

[4 marks]

Question 3

Use proof by contradiction to show there are no integers a and b such that,

$$10a + 15b = 17$$

[3 marks]

Question 4

A-Level Examination Question from June 2022, Paper 1, Q7 (Edexcel)

(i) Given that p and q are integers such that,

$$pq \text{ is even}$$

use algebra to prove by contradiction that at least one of p or q is even.

[3 marks]

(ii) Given that x and y are integers such that,

- $x < 0$
- $(x + y)^2 < 9x^2 + y^2$

show that $y > 4x$

[2 marks]

Question 5

A-Level Examination Question from October 2021, Paper 1, Q15 (Edexcel)

(i) Use proof by exhaustion to show that for $n \in \mathbb{N}, n \leqslant 4$

$$(n+1)^3 > 3^n$$

[2 marks]

(ii) Given that $m^3 + 5$ is odd, use proof by contradiction to show, using algebra, that m is even.

[4 marks]

Question 6

A-Level Examination Question from October 2020, Paper 1, Q16 (Edexcel)

Prove by contradiction that there are no positive integers p and q such that,

$$4p^2 - q^2 = 25$$

[4 marks]

Question 7

(i) Prove by contradiction that if n^2 is a multiple of 3, then n is a multiple of 3

Hint : Consider numbers in the form $3k + 1$ and $3k + 2$

[3 marks]

(ii) Hence prove by contradiction that $\sqrt{3}$ is an irrational number.

[6 marks]

Question 8

Let x be an irrational number and y be a rational number.
Use proof by contradiction to show that $x + y$ is irrational.

[6 marks]